Dynamics of Parabolic Transcendental Entire Functions

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by

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Contents

Chapter 1

Introduction

The study of the dynamics of holomorphic maps was started by Pierre Fatou and Gaston Julia in the early twentieth century. They initially studied the behaviour of rational maps (includes polynomials) $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ under iteration. In their study, they showed that the Riemann sphere splits into two disjoint completely invariant sets. The set of normality is known today as the *Fatou set*, and is defined as the set of points $z \in \widehat{\mathbb{C}}$ such that $(f^n(z))_{n\geq 0}$ is a normal family in a neighbourhood of z. The other set is its complement and called the *Julia set*. We will denote the *Fatou set* by $\mathcal{F}(f)$ and the *Julia set* by $\mathcal{J}(f)$.

Fatou was the first mathematician who considered the dynamics of *transcendental entire functions* in 1926. He observed that the dynamics of rational and transcendental maps share some of the basic features. He also studied the dynamics of the transcendental function $S_\lambda\colon z \mapsto \lambda\sin(z)$ and noticed that $\mathcal{J}(S_\lambda)$ contains infinitely many curves on which the iterates converge to infinity. A huge contribution to the study of the dynamics of transcendental entire functions was made by Baker. One of the important results he proved states that the Julia set is the closure of the set of *repelling periodic* points [Bak68]. In 1985, Sullivan proved a crucial result in the field. He showed that there are no *wandering* components in the Fatou set of a rational map [Sul85].

After that, the topic received increasing attention from researchers which continue until this day. In particular, researchers started to investigate the dynamics of specific classes of transcendental entire functions. For instance, Misiurewicz answered a question by Fatou when he proved that the Julia set of the exponential map $z \mapsto \exp(z)$ is the whole complex plane [Mis81]. Devaney and Tangerman studied the geometry of the Julia set of the map $E_\lambda \colon z \mapsto \lambda \exp(z)$ for $0 < \lambda \leq 1/e$ [DT86]. In fact, they observed that Cantor bouquet Julia sets occur in other classes of transcendental functions, such as S_λ for $0 < \lambda < 1$ and $z \mapsto \lambda \cos(z)$ for $|\lambda| < 1$.

Although the dynamical behaviour of polynomials and transcendental entire functions have similarities, there also are major differences. The dynamics of polynomials has a special characteristic since infinity is a *superattracting fixed* point for all polynomials. If $P \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a polynomial of degree $d \geq 2$, then $P(\infty) = \infty$ and the *immediate attracting basin* of infinity is given by

$$A(\infty) := \{z \in \widehat{\mathbb{C}} \colon P^n(z) \to \infty \text{ as } n \to \infty\}.$$

The *filled in Julia set* $K(P)$ is the set of all $z \in \mathbb{C}$ for which the orbit of z under P is bounded. Hence, $A(\infty) = \widehat{\mathbb{C}} \backslash K(P)$. If the filled in Julia set $K(f)$ contains all the finite critical points of P then the Julia set $\mathcal{J}(P)$ is connected [Mil06, Theorem 9.5]. This means that $A(\infty)$ is simply connected for such polynomials. By the *Riemann mapping theorem*, there is a conformal isomorphism $\phi \colon \mathbb{D} \to A(\infty)$ which conjugates $z \mapsto z^d$ to P on $A(\infty)$.

Suppose that $K(P)$ (or equivalently $\partial K(P) = \partial A(\infty)$) is locally connected. Then by the *Carathéodory-Torhorst theorem*, the map ϕ has a surjective continuous extension from the boundary of the unit disc to the boundary of $A(\infty)$. Note that $\mathcal{J}(P) = \partial A(\infty)$ by a result of Fatou and Julia. So this model of a pinched disc gives a complete description of the dynamics of P. It conjugates $z \mapsto z^d$ on the unit circle to the function P on its Julia set. For this reason, it was very crucial to know when the Julia set of a polynomial is locally connected.

Local connectivity of the Julia set was proved for several classes of polynomials. The simplest polynomials are those that are *hyperbolic*, i.e the orbits of all critical points tend to attracting orbits, see [Mil06, Theorem 19.2]. Another class of polynomials for which locally connectivity of the Julia set was proved is *subhyperbolic*, i.e the orbit of each critical point is either finite or converges to an attracting periodic orbit, see [Mil06, Theorem 19.7]. It was also proved that the connected Julia set of a *geometrically finite* polynomial (the orbit of each critical point in its Julia set is preperiodic) is locally connected [CG93, Thorem 4.3]. For more classes with locally connected Julia sets, see [Mil00]. Authors were interested in proving this property of the Julia set of polynomials because it implies simple topological dynamics. However, this property of the Julia set does not have the same implication in the dynamics of transcendental maps as in the polynomial case. For example, the Julia set of the exponential map $\exp(z)$ is the whole complex plane, which is trivially locally connected, but one can draw no conclusion from this. In fact, some questions about the dynamics of this specific function remain open to this day, see [SRG15, section 7]. The reason for this behaviour is that infinity is an essential singularity for transcendental functions, and hence not a super-attracting fixed point.

In spite of this fact, a similar technique can be used to study the Julia set of some classes of transcendental functions. We will describe this idea in detail. For a transcendental function $f\colon \mathbb{C} \to \mathbb{C}$, the set of escaping points is given by:

$$I(f) := \{z \in \mathbb{C}\colon f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The set of escaping points plays an important role in the dynamics of transcendental functions. Eremenko studied in [Ere89] the set of escaping points for any transcendental entire function f. He proved significant properties for the set of escaping points $I(f)$.

Theorem 1.1. [Ere89] *Let f be a transcendental entire function. Then $I(f)$ has the following properties.*

(a) $\mathcal{J}(f) = \partial I(f)$.

(b) $I(f)$ *is nonempty.*

(c) $I(f) \cap \mathcal{J}(f) \neq \emptyset$.

The result $I(f) \cap \mathcal{J}(f) \neq \emptyset$ is the same as for polynomials, but with a completely different proof. The set of escaping points plays a particularly important role in the dynamics of transcendental entire functions in class $\mathcal{B}$, for which the set of singular values is bounded. It was also proved by Eremenko and Lyubich that the Julia set of such functions is the closure of the set of escaping points, see [Ere89, Corollary] and [EL92, Corollary].

An entire function is called *disjoint type* if it is hyperbolic and the orbits of all singular values tend to one attracting fixed point. The following result was proved by Rempe-Gillen in [Rem09].

Theorem 1.2. *Let $f \in \mathcal{B}$ be hyperbolic, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is of disjoint-type. Then there exists a continuous surjection $\phi\colon \mathcal{J}(g) \to \mathcal{J}(f)$, such that $f(\phi(z)) = \phi(g(z))$ for all $z \in \mathcal{J}(g)$.*

Furthermore, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

Then Mihaljevic′-Brandt generalized this result to a larger class of trancseden-tal entire function which is the class of *strongly subhyperbolic* functions, see [MB12].

The goal of this thesis is to explore a similar theory as in [Rem09] and [MB12] for a class of transcendental entire functions we called *parabolic*. We call a transcendental entire function f with bounded set of singular values *parabolic* if the set of the singular values of f is in the Fatou set, and the intersection of the postsingular set and the Julia set of f is finite and not empty, and it contains only the parabolic periodic points of f.

The notation of *parabolic* rational maps already exists, see [DU91a] and [DU91b]. A rational map R is *parabolic* if the Julia set $\mathcal{J}(R)$ contains no critical points of R and the set of parabolic periodic points of R is not empty. The definition of parabolic meromorphic functions was given in [Zhe11] and it is similar to our definition.

We will give below some examples of *parabolic* entire functions functions.

Example (1). The function $f_1 \colon z \mapsto \exp(z) - 1$ is *parabolic.* It has a unique parabolic fixed point at $z = 0$ with multiplicity 2 and multiplier one. So there is only one immediate parabolic basin of zero U_0 which is mapped into itself under f_1. The function f_1 has no critical values and only one asymptotic value which is -1, and hence the set of singular values of f_1 is given by $S(f_1) = \{-1\}$. Every immediate parabolic basin contains a singular value by [Ber93, Theorem 7]. Thus, the singular value -1 must belong to the set U_0 which is contained in $\mathcal{F}(f_1)$. Hence, the set $S(f_1)$ is contained in the Fatou set of f_1.

Example (2). The function $f_2 \colon z \mapsto \sin(z)$ is *parabolic.* It has a unique parabolic fixed point at $z = 0$ with multiplicity 3 and multiplier one. Hence, there are two immediate parabolic basins of zero such that each basin is mapped into itself under f_2. The set of singular values of f_2 is $S(f_2) = \{-1, 1\}$ where -1 and 1 are the critical values of f_2. Each singular value belongs to one immediate parabolic basin of 0. This means that $S(f_2) \subset \mathcal{F}(f_2)$.

The following example was mentioned in [ES18].

Example (3). The function $f_3\colon z \mapsto z^2\exp(z - z^2)$ is *parabolic*. Observe that

$$f_3(z) = e + (z-1) + (z-1)^2 + O\left((z-1)^3\right).$$

This function has a parabolic fixed point at $z = 1$ with multiplicity 2 and multiplier one, and a superattracting point at zero. So there is only one immediate parabolic basin of 1, and an immediate attracting basin of zero, say U_1 and U_2. The set of singular values of f_3 is given by

$$S(f_3) := \left\{0, f_2((1+\sqrt{17})/4), f_2((1-\sqrt{17})/4)\right\}.$$

Let $w_1 := f_3((1+\sqrt{17})/4)$ and $w_2 := f_3((1-\sqrt{17})/4)$. Note that $0 \in U_2$. By looking at the graph of the function $x \mapsto f_3(x)$ where $x \in \mathbb{R}$, we see that the orbit of w_1 is attracted to 1, and the orbit of w_2 is attracted to zero. This means that $w_1 \in U_1$ and $w_2 \in U_2$. Hence, $S(f_3) \subset \mathcal{F}(f_2)$ because U_1 and U_2 are components of the Fatou set of f_2.

Our main result for parabolic transcendental entire maps is the following.

Theorem 1.3. *Let f be a parabolic transcendental entire function, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is of disjoint type.Then there exists a continuous surjection $\phi\colon \mathcal{J}(g) \to \mathcal{J}(f)$, such that $f(\phi(z)) = \phi(g(z))$ for all $z \in \mathcal{J}(g)$.*

Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

A hyperbolic function is uniformly expanding on its Julia set with respect to a suitable hyperbolic metric. This property is a crucial ingredient in the proof of local connectivity of the Julia set of such a function if it is a polynomial [Mil06, Theorem 19.2], and in the proof of Theorem 1.2.

For parabolic functions, this approach cannot work without modification. This is because we we cannot include the parabolic periodic points in the domain on which such an expanding hyperbolic metric is defined. For polynomials, Douady and Hubbard overcame this obstacle by modifying the hyperbolic metric near all parabolic periodic points, in such a way that the new metric become expanding, see [DH85] and [CG93, Thorem 4.3] and the construction preceding it. For the new metric, the expansion is weaker near a parabolic point but sufficient for the proof of local connectivity to proceed.

The general strategy of our proof of Theorem 1.3 is to combine the same idea of modifying the metric with the proof of Theorem 1.2. There are a number of difficulties to overcome in this approach. We will briefly discuss the two main ones.

Firstly, a key point in the construction of the expanding metric for parabolic polynomials is that the full preimage of a small neighbourhood of a parabolic point is bounded, see [CG93, page 95]. Naturally, this is not the case for transcendental entire functions. Instead, we develop in Theorem 4.10 an alternative argument using hyperbolic geometry.

Secondly, the existence of a semiconjugacy in the polynomial case usually uses the Carathéodory-Torhorst theorem in a nontrivial manner. Indeed, in the usual proof they first pass to an iterate f^n of a polynomial f for some $n \in \mathbb{N}$, for which all parabolic points are fixed and of multiplier one. Then, they construct an expanding metric for this iterate and prove that the Julia set of this iterate is locally connected. Since $\mathcal{J}(f) = \mathcal{J}(f^n)$ then the Julia set of f is locally connected, and hence there exists a semiconjugacy between $z \mapsto z^d$ and f where d is the degree of f.

In the transcendental setting, it is less clear whether Theorem 1.3 for f^n implies the same statement for the function f. Therefore, we proceed as follows. We first construct the desired expanding metric for the iterate f^n, and then we deduce that the same metric has a suitable expansion property for the original function f, see Proposition 4.11 and Propositions 4.12. With there propositions, we can prove Theorem 1.3 directly for f.

Chapter 2

Preliminaries

2.1 Notation

In this section we will give all notations that we use in this thesis. We denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\widehat{\mathbb{C}}$.

For a disc in the complex plane centered at $a \in \mathbb{C}$ and with radius r, we use the notation

$$D(a, r) = \{z \in \mathbb{C} \colon |z - a| < r\}.$$

For a right half plane in the complex plane we define

$$\mathbb{H}_r(R) := \{z \in \mathbb{C} \colon \mathrm{Re} > R\},$$

and for a left half plane in the complex plane we define

$$\mathbb{H}_l(R) := \{z \in \mathbb{C} \colon \mathrm{Re} < R\}.$$

For special domains in the complex plane we will use the following notations. We denote the unit disc by

$$\mathbb{D} := D(0, 1),$$

the punctured unit disc by

$$\mathbb{D}^* := D(0,1)\backslash\{0\},$$

the right half plane by

$$\mathbb{H}_r := \mathbb{H}_r(0),$$

and the left half plane by

$$\mathbb{H}_l := \mathbb{H}_l(0).$$

For a subset $A \subset \mathbb{C}$ the set of boundary points and the closure of A in the complex plane will be denoted by ∂A and $\overline{A}$, respectively.

The Euclidean distance between the set $A \neq \emptyset$ and a point $c \in \mathbb{C}$ is given by

$$\operatorname{dist}(A, c) := \inf\{|z - c| \colon z \in A\}.$$

2.2 Hyperbolic metric

In this section we shall refer to [BM07]. A domain U is called *hyperbolic* if its complement in the complex plane contains at least two points. An important model of the hyperbolic plane is the unit disk $\mathbb{D}$ with the hyperbolic metric that is given by:

$$\rho_{\mathbb{D}}(z)|\mathrm{d}z| = \frac{2|\mathrm{d}z|}{1-|z|^2},$$

where $\rho_{\mathbb{D}} \colon \mathbb{D} \to (0, \infty)$ with $\rho_{\mathbb{D}}(z) = 2/(1-|z|^2)$ is called the density of the hyperbolic metric.

For a simply connected domain $U \subset \mathbb{C}$ *the Riemann Mapping Theorem* states that there exists a conformal map from the domain U onto $\mathbb{D}$. Then we define *the hyperbolic metric on* U as follows.

Definition 2.1. [BM07, Definition 6.2] *Suppose that ϕ is a conformal map of a simply connected domain U onto $\mathbb{D}$. Then the density of the hyperbolic metric of U is defined as*

$$\rho_U(z) = \rho_{\mathbb{D}}(\phi(z))|\phi'(z)|.$$

Let us justify that ρ_U in Definition 2.1 is independent of the choice of the conformal map ϕ. Suppose that ψ is another conformal map from the domain U onto $\mathbb{D}$. Then $\psi = \mathcal{M} \circ \phi$ where $\mathcal{M}$ is a Möbius self map of $\mathbb{D}$. The hyperbolic metric $\rho_{\mathbb{D}}(z)|\mathrm{d}z|$ is invariant under $\mathcal{M}$, i.e. $\rho_{\mathbb{D}}(z) = \rho_{\mathbb{D}}(\mathcal{M}(z))|\mathcal{M}'(z)|$ for all $z \in \mathbb{D}$. Hence, we have

$$\begin{aligned} \rho_{\mathbb{D}}(\psi(z))|\psi'(z)| &= \rho_{\mathbb{D}}(\mathcal{M}(\phi(z)))|\mathcal{M}'(\phi(z))||\phi'(z)| \\ &= \frac{\rho_{\mathbb{D}}(\phi(z))}{|\mathcal{M}'(\phi(z))|}|\mathcal{M}'(\phi(z))||\phi'(z)| = \rho_{\mathbb{D}}(\phi(z))|\phi'(z)|. \end{aligned} \tag{2.1}$$

Example. By Definition 2.1 where $\phi\colon \mathbb{H}_r \to \mathbb{D}$ is given by $z \mapsto (1-z)/(1+z)$, we can calculate the hyperbolic metric on the *right half plane* $\mathbb{H}_r$.

$$\begin{aligned} \rho_{\mathbb{H}_r}(z) &= \frac{2|\phi'(z)|}{1-|\phi(z)|^2} = \frac{2|1+z|^2}{|1+z|^2-|1-z|^2}.\frac{2}{|1+z|^2} \\ &= \frac{4}{(1+\mathrm{Re}z)^2+(\mathrm{Im}z)^2-(1-\mathrm{Re}z)^2-(\mathrm{Im}z)^2} = \frac{1}{\mathrm{Re}z}. \end{aligned} \tag{2.2}$$

More generally, we can define a hyperbolic metric also on a multiply connected hyperbolic domain. By *the Planar Uniformisation Theorem* [Mil06, Theorem 10.2], if $\Omega \subset \mathbb{C}$ is a hyperbolic domain then there exists a holomorphic covering map $\phi\colon \mathbb{D} \to \Omega$. The next result shows that the hyperbolic metric of $\mathbb{D}$ can be transferred to a hyperbolic metric on Ω. In particular, if the map ϕ is known then we can use it to compute the hyperbolic metric on Ω explicitly.

Theorem 2.2. [Mil06, Theorem 10.3] *Let $\phi\colon \mathbb{D} \to \Omega$ be a holomorphic uni-*

versal covering. Then there is a unique metric $\rho_\Omega(w)|\mathrm{d}w|$ such that

$$\rho_{\mathbb{D}}(z) = \rho_\Omega(\phi(z))|\phi'(z)|.$$

The hyperbolic metric on Ω in Theorem 2.2 is independent of the choices of both the inverse branch of ϕ that was chosen to define the metric, and the covering map. If $\psi\colon \mathbb{D} \to \Omega$ is another covering map of Ω then there is a Möbius self map $\mathcal{M}$ of $\mathbb{D}$ such that $\psi = \mathcal{M} \circ \phi$. So to prove that the hyperbolic metric on Ω is defined independently of the covering map ϕ, we can do the same as in (2.1).

Let us justify the independence of the hyperbolic metric on Ω of the choice of the inverse branch of ϕ. Let U_1 and U_2 be two simply connected subsets of Ω such that $U_1 \cap U_2 \neq \emptyset$. Suppose that ψ_1 and ψ_2 are two inverse branches of ϕ such that $\psi_1\colon U_1 \to \mathbb{D}$ and $\psi_2\colon U_2 \to \mathbb{D}$. Then there exists a Möbius self map $\mathcal{M}$ of $\mathbb{D}$ such that $\psi_2 = \mathcal{M} \circ \psi_1$ on $U_1 \cap U_2$. Thus, by the invariance of the hyperbolic metric under $\mathcal{M}$ we have

$$\begin{aligned}\rho_{\mathbb{D}}(\psi_2(z))|\psi_2'(z)| &= \rho_{\mathbb{D}}\left(\mathcal{M}\left(\psi_1(z)\right)\right)|\mathcal{M}'(\psi_1(z))||\psi_1'(z)| \\ &= \rho_{\mathbb{D}}(\psi_1(z))|\psi_1'(z)|.\end{aligned}$$

Now we can define the hyperbolic metric of any hyperbolic domain $\Omega \subset \mathbb{C}$ as follows.

Definition 2.3. *Let $\Omega \subset \mathbb{C}$ be a hyperbolic domain. Then the unique metric given by Theorem 2.2 is called the hyperbolic metric of Ω.*

If $U \subset \mathbb{C}$ is a hyperbolic domain and $z, w \in U$ then we denote the *hyperbolic length* of a smooth curve γ joining z and w by $\ell_U(\gamma)$. This length is defined as follows:

$$\ell_U(\gamma) = \int_\gamma \rho_U(z)|\mathrm{d}z|. \tag{2.3}$$

Then the *hyperbolic distance* $\mathrm{d}_U(z, w)$ between $z, w \in U$ is given by

$$\mathrm{d}_U(z, w) = \inf_{\gamma} \ell_U(\gamma), \tag{2.4}$$

where the infimum is taken over all the smooth curves in U that join z and w.

Remark. The hyperbolic domain U with the hyperbolic distance d_U is a complete metric space.

We define the *hyperbolic derivative* of a holomorphic function $f\colon V \to W$ with respect to the hyperbolic metric as follows:

$$\|\mathrm{D}f(z)\|_V^W := \frac{\rho_W(f(z))}{\rho_V(z)}|f'(z)|.$$

If $V \subset W$, let $\imath\colon V \to W$ be the inclusion map. Then we define $\|\mathrm{D}f(z)\|_W^W$ as follows:

$$\begin{aligned}\|\mathrm{D}f(z)\|_W^W = \frac{1}{\|\mathrm{D}\imath(z)\|_V^W}\|\mathrm{D}f(z)\|_V^W &= \frac{\rho_V(z)}{\rho_W(z)}\frac{\rho_W(f(z))}{\rho_V(z)}|f'(z)| \\ = \frac{\rho_W\left(f(z)\right)}{\rho_W(z)}|f'(z)|.\end{aligned}$$

Set $\|\mathrm{D}f(z)\|_W := \|\mathrm{D}f(z)\|_W^W$.

We refer to the following result as Pick's Theorem. It establishes important properties of the hyperbolic metric. It is also called the Schwarz-Pick Lemma, see [BM07, Theorem 10.5] and [Mil06, Theorem 2.11].

Theorem 2.4. (Pick's Theorem) *Let $f\colon V \to W$ be a holomorphic function between two hyperbolic domains. Then the following hold*

(a) *f does not increase the hyperbolic metric; i.e. for all $z \in V$ we have*

$$\rho_W(f(z))|f'(z)| \le \rho_V(z),$$

or equivalently

$$\|\mathrm{D}f(z)\|_V^W \leq 1;$$

(b) *for any $z \in V$, we have $\|\mathrm{D}f(z)\|_V^W = 1$ if and only if f is a covering map;*

(c) *if $V \subsetneq W$ then $\rho_V(z) > \rho_W(z)$ for all $z \in V$.*

Example. The hyperbolic metric on $\mathbb{D}^*$ can be calculated by using the holomorphic covering map $\phi\colon \mathbb{H}_r \to \mathbb{D}^*$ given by $z \mapsto \exp(-z)$. It follows from Theorem 2.4(b) that the hyperbolic metric of $\mathbb{D}^*$ is the unique metric whose pullback under ϕ is the hyperbolic metric of $\mathbb{H}_r$. Let $w = \phi(z)$ for $z \in \mathbb{H}_r$, and we have

$$\rho_{\mathbb{D}^*}(w) = \rho_{\mathbb{H}_r}(z)/|\phi'(z)| = \frac{\exp(\mathrm{Re}z)}{\mathrm{Re}z} = \frac{1}{|w|\log(1/|w|)}.$$

Example. We can use the hyperbolic metric of $\mathbb{D}^*$ and Theorem 2.4(b) again to compute the hyperbolic metric on the set $U := \mathbb{C}\backslash\overline{D(0,R)}$. Note that the function $\psi\colon U \to \mathbb{D}^*$ with $z \mapsto R/z$ is a covering map. Hence, we have

$$\rho_U(z) = \rho_{\mathbb{D}^*}(\psi(z))|\psi'(z)| = \frac{|z|}{R\log(|z|/R)}.(R/|z|^2) = \frac{1}{|z|\log(|z|/R)}. \quad (2.5)$$

The next Theorem can be proved by Schwarz Lemma and Koebe's Theorem, and it gives useful estimates on the hyperbolic density of the hyperbolic metric.

Theorem 2.5. [Mil06, Corollary A.8] *Let $U \subset \mathbb{C}$ be a simply connected hyperbolic domain. Then the hyperbolic density ρ_U satisfies*

$$\frac{1}{2\mathrm{dist}(z,\partial U)} \leq \rho_U(z) \leq \frac{2}{\mathrm{dist}(z,\partial U)}.$$

2.3 An introduction to holomorphic dynamics

Let $U \subset \mathbb{C}$ and $f\colon U \to \mathbb{C}$ be a holomorphic function. We denote the nth iterate of f by f^n. If $z \in \mathbb{C}$ then we call the sequence $(f^n(z))_{n\geq 0}$ the forward orbit of z. If $z_0 \in U$ and $f(z_0) = z_0$ then we call z_0 a *fixed point* of f. We call z_0 a *periodic point* of f if there is $n \in \mathbb{N}$ such that $f^n(z_0) = z_0$, and the set $\{z_0, f(z_0), \ldots, f^{n-1}(z_0)\}$ the periodic cycle of z_0. If n is the smallest number with this property then we call n the *period* of z_0. If $f^m(z_0)$ is periodic for some $m \geq 1$ but z_0 is not periodic then we call z_0 a *strictly preperiodic point of* f.

The *multiplier of* z_0 is defined to be $\lambda := (f^n)'(z_0)$. We distinguish periodic points according to the multiplier λ. If $0 \leq |\lambda| < 1$ then z_0 is an *attracting periodic point* of f. For the special case $\lambda = 0$ the point z_0 is called *superattracting point* of f. If $|\lambda| = 1$ then z_0 is an *indifferent periodic point of* f and in this case $\lambda = e^{2\pi i\theta}$ for some θ. The point z_0 is called *rationally indifferent point* or *parabolic point* if θ is rational, and called *irrationally indifferent point* or *Cremer point* if θ is irrational. We denote the set of all *parabolic periodic points* of f by $\mathrm{Par}(f)$.

Let $\Xi := \{z_0, z_1, \ldots, z_{n-1}\}$ be an attracting periodic cycle of period n. We define *the attracting basin* of z_0 to be the open set $\mathcal{A}(z_0) \subset \mathbb{C}$ which consists of all points $z \in \mathbb{C}$ for which the successive iterates converge to some point in Ξ. Then *the immediate attracting basin* of z_0 denoted by $\mathcal{A}_0$ is the connected component of $\mathcal{A}(z_0)$ that contains z_0.

Suppose that z_0 is a parabolic fixed point with multiplier one. We will give in the next section the definition of an *attracting vector* and explain the convergence of an orbit in the direction of an *attracting vector*. We will also give the definition of the *parabolic basin of attraction* that is associated to each attracting vector at z_0.

It it known that all attracting cycles for a holomorphic map f are in the Fatou set, but all repelling cycles for f are in the Julia set.

Lemma 2.6. [Mil06, Lemma 4.6] *Every attracting periodic point is in the Fatou set $\mathcal{F}(f)$. In fact, the entire attracting basin for an attracting periodic point is in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

We denote by A_{Att} the set of all points whose orbits converge to attracting cycles, and by A_{Par} the set of all points whose orbits converge nontrivially to a parabolic cycle, where the parabolic points themselves are not in A_{Par}. It follows from Lemma 2.6 and Lemma 3.16 that the sets A_{Att} and A_{Par} are contained in the Fatou set.

We will now define *the set of singular values* of f. This set has a significant impact on the dynamics of a holomorphic function f. A point $w \in \mathbb{C}$ is a *critical value* if it is the image under f of a critical point of the function f. Denote the set of all *critical values* of f by $CV(f)$. We call $a \in \mathbb{C}$ an *asymptotic value* of f if there exists a curve $\gamma : (0, \infty) \to \mathbb{C}$ with the property $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a$. Denote the set of all *asymptotic values* of f by $AV(f)$. Note that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a polynomial then $AV(f) = \emptyset$ because ∞ is a superattracting point of f.

Now, we define the *set of singular values* to be

$$S(f) := \overline{CV(f) \bigcup AV(f)}.$$

Then the *postsingular set* of f is given by

$$P(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}.$$

We denote by $P'(f)$ the set of all finite limit points of the set $P(f)$.

The following result was proved in [Mil06] for rational maps but the proof can be carried over for transcendental maps.

Theorem 2.7. [Mil06, Theorem 11.7] *Every Cremer fixed or periodic point for f is in the postsingular set $P(f)$.*

The *Eremenko-Lyubich class* $\mathcal{B}$ is the class of all transcendental entire functions for which the set $S(f)$ is bounded. The following theorem was proved by Eremenko and Lyubich for functions in class $\mathcal{B}$.

Theorem 2.8. [EL92, Theorem 1] *Let $f \in \mathcal{B}$. If $z \in \mathcal{F}(f)$ then the orbit $(f^n(z))_{n\geq 0}$ does not tend to ∞.*

Equivalent result to Theorem 2.8 was anounced earlier in [Ere89, Theorem 4]. The following result is due to Eremenko and Lyubich, see [Ere89, Corollary] and [EL92, Corollary]. In fact, it is an immediate consequence of Theorem 1.1(a) and Theorem 2.8.

Theorem 2.9. *If $f \in \mathcal{B}$ then $\mathcal{J}(f) = \overline{I(f)}$.*

2.4 The Components of the Fatou set

Let U be a component of the Fatou set $\mathcal{F}(f)$. Since $\mathcal{F}(f)$ is completely invariant under f then $f^n(U)$ must be contained in one component of $\mathcal{F}(f)$. If U has the property that $f^m(U) \cap f^n(U) = \emptyset$ for all $m > n \geq 0$ then U is called a *wandering domain*. It was proved by Sullivan that rational maps have no wandering domains, see [Sul85] and [Mil06, Theorem 16.4].

However, for transcendental maps there are several examples of entire functions that were constructed and proven to have wandering domains [Ber93, p. 168]. There are also some classes known to have no wandering domains.

One example is the class $\mathcal{S}$ of all entire functions with finite set of singular values [EL92, Theorem 3]. If U is a wandering domain and $z \in U$ then it is well-known that all limit functions of $(f^n(z))_{n\geq 0}$ are constant, see [Fat20], [Cre32] and [BHK$^+$93, p. 370]. However, Baker proved that all constant limit functions in a component of $\mathcal{F}(f)$ not necessarily wandering belong to $P(f) \cup \{\infty\}$ [Bak70].

We say U is a *periodic Fatou component* if it has the property $f^n(U) \subset U$ for some $n \in N$, and the smallest n with this property is called the *period* of U. If $f^m(U)$ is periodic for some $m \in \mathbb{N}$ and U is not periodic then we call U a *strictly preperiodic* component.

A classification of the periodic components of the Fatou set was essentially given by Fatou [Fat20] and Cremer [Cre32]. For more information on the history of this classification we refer to [Ber93, Theorem 6] and the discussion following it.

In the following result we will give the classification of the periodic components of the Fatou set for an entire function.

Theorem 2.10. *Let f be an entire function and let U be a periodic Fatou component of period k. Then one of the following holds.*

(a) *U contains an attracting periodic point z_0 of period k, and $f^{kn}(z) \to z_0$ for $z \in U$ as $n \to \infty$. In this case U is called the immediate attracting basin of z_0.*

(b) *∂U contains a periodic point z_0 of period k with $(f^k)'(z_0) = 1$, and $f^{kn}(z) \to z_0$ for $z \in U$ as $n \to \infty$. In this case U is called the immediate parabolic basin of z_0 or Leau domain.*

(c) *There exists an analytic homeomorphism $\phi\colon U \to \mathbb{D}$ such that $\phi\left(f^k\left(\phi^{-1}\left(z\right)\right)\right) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, U is called a Siegel disc.*

(d) *There exists* $z_0 \in \partial U$ *such that* $f^{kn}(z) \to z_0$ *for* $z \in U$ *as* $n \to \infty$, *but* z_0 *is an essential singularity of* f^k. *In this case* U *is called a Baker domain.*

A *preperiodic* Fatou component is a preimage of a component of one of the types above. If f is a transcendental entire function then all preperiodic components of the Fatou set of f are simply connected [Ber93, Theorem 9]. If U is a multiply-connected component of $\mathcal{F}(f)$ then U is a wandering domain and $U \subset I(f)$ [Bak84, Theorem 3.1]. This result together with [EL92, Theorem 1] implies that if $f \in \mathcal{B}$ then all components of the Fatou set of f are simply connected.

Note that Baker domains cannot occur for rational maps. Therefore, every component of the Fatou set of a polynomial is either an attracting component, a parabolic component, a Siegel disc or a preimage of such a component.

There is a strong relation between the set of singular values $S(f)$ and the components of the Fatou set $\mathcal{F}(f)$. This is stated in the next theorem.

Theorem 2.11. [Ber93, Theorem 7] *Let* f *be an a non-linear entire function, and let* $\Omega = \{U_0, U_1, \ldots, U_{k-1}\}$ *be a periodic cycle of components of* $F(f)$. *Then the following hold.*

(a) *If* Ω *is a cycle of immediate attracting or parabolic basins then* $U_j \cap S(f) \neq \emptyset$ *for some* $j \in \{0, 1, \ldots, k-1\}$. *More precisely, the intersection* $U_j \cap S(f)$ *contains a point that is not preperiodic or* U_j *contains a periodic critical point and* Ω *is a cycle of superattracting basins.*

(b) *If* Ω *is a cycle of Siegel discs then* $\partial U_j \subset P(f)$ *for all* $j \in \{0, 1, \ldots, k-1\}$.

Chapter 3

Parabolic points and Fatou coordinates

In this chapter we will study the local dynamics of a holomorphic map near a parabolic periodic point. We refer the reader to [Mil06] and [Bea91].

3.1 Attracting and repelling vectors

Suppose that f is a nonlinear function that is holomorphic in a neighbourhood of zero, and satisfies $f(0) = 0$ and $f'(0) = \lambda$ where $\lambda \in \mathbb{C}$. Recall from the previous section that if $\lambda = e^{2\pi i m/n}$ where $m, n \in \mathbb{N}$ and $(m, n) = 1$ then f has a parabolic fixed point at zero. For simplicity, we can assume that $\lambda = 1$ because otherwise the iterate f^n has a fixed point at zero with multiplier one. For $a \neq 0$ and $p \geq 1$ we can write

$$f(z) = z + az^{p+1} + O(z^{p+2}), \tag{3.1}$$

The number $p + 1$ is called the *multiplicity* of the fixed point. We will give below the definition of attracting and repelling vectors as in [Mil06, Definition on page 104].

Definition 3.1 (Attracting and repelling vectors at zero). *Let f be as in (3.1). A complex number $\mathbf{v}$ is called a repelling vector for f at zero if $pa\mathbf{v}^p = 1$, and an attracting vector if $pa\mathbf{v}^p = -1$.*

Remark. (1) The vector $\mathbf{v}$ here should be thought of as a tangent vector to $\mathbb{C}$ at zero. For example, as a tangent vector to the curve $t \mapsto \mathbf{v}t$ at $t = 0$.

(2) The definition of attracting vectors implies that there are p equally spaced attracting vectors at zero. Similarly, there are p equally spaced repelling vectors at zero which alternate with the attracting vectors at zero.

(3) The repelling vectors of f are attracting vectors of a local inverse of f in a neighbourhood of zero.

Definition 3.2. *Let f be as in (3.1). Suppose that $\mathbf{v}$ is an attracting vector at zero. If a point z has the property that $n^{1/p} f^n(z) \to \mathbf{v}$ as $n \to \infty$ then we say that the orbit $(f^n(z))_{n \geq 0}$ tends to zero from the direction $\mathbf{v}$.*

The following Lemma describes the local dynamics of a function near its parabolic fixed point.

Lemma 3.3. [Mil06, Lemma 10.1] *Let f be as in (3.1). If $z \in \mathbb{C}$ and its orbit $(f^n(z))_{n \geq 0}$ converges to zero nontrivially, i.e. $f^k(z) \neq 0$ for all $k \geq 0$, then there exists an attracting vector $\mathbf{v}$ at zero such that the orbit of the point z tends to zero from the direction $\mathbf{v}$.*

Now, suppose that f has a parabolic fixed point $\zeta \in \mathbb{C}$ with multiplier one. Then we conjugate f to a map g such that

$$g(w) = f(w + \zeta) - \zeta,$$

The function g has a parabolic fixed point at zero with multiplier one. Thus, if we define any attracting(repelling) vector $\mathbf{v}$ for g at zero as in Definition 3.4. Then the same vector $\mathbf{v}$ is an attracting(repelling) vector for f at ζ.

Definition 3.4 (Attracting and repelling vectors at a fixed point)**.** *Let f be an entire function that has a parabolic fixed point at $\zeta \in \mathbb{C}$ with multiplier one. A complex number $\mathbf{v}$ is called an attracting vector for f at ζ if it is an attracting vector for $g = \phi \circ f \circ \phi^{-1}$ at zero where $\phi\colon z \to z - \zeta$. Similarly, for a repelling vector for f at ζ.*

By conjugating f to g again and applying Lemma 3.3 to g, the following Lemma is then immediate.

Lemma 3.5. *Suppose that f has a parabolic fixed point at $\zeta \in \mathbb{C}$ with multiplier one and multiplicity $p_\zeta + 1$. If $z \in \mathbb{C}$ and its orbit $(f^n(z))_{n\geq 0}$ converges to ζ nontrivially then there exists an attracting vector $\mathbf{v}$ at ζ such that $n^{1/p_\zeta}(f^n(z) - \zeta) \to \mathbf{v}$ as $n \to \infty$, and we say that the orbit of z tends to ζ from the direction $\mathbf{v}$.*

The next Lemma gives properties of a function and its derivative near a parabolic fixed point $\zeta \in \mathbb{C}$ with multiplier one. We are going to use these properties in the proof of Lemma 4.7.

Lemma 3.6. *Let f be as in Lemma 3.5. Let $\mathbf{v}$ be a repelling vector at the parabolic fixed point ζ. Then there exists $\epsilon > 0$ such that*

$$|f(z) - \zeta| > |z - \zeta|,$$

and the derivative of f satisfies

$$|f'(z)| > 1,$$

for all z *with* $0 < |z - \zeta| < \epsilon$ *and* $|\mathrm{Arg}\left((z-\zeta)/\mathbf{v}\right)| < \pi/(4p_\zeta)$ *where* $-\pi < \mathrm{Arg}(z) \leq \pi$.

Proof. Let us first assume that f has a parabolic fixed point at zero with multiplier one and multiplicity $p+1$, i.e. f can be written as in (3.1). Then, we have

$$\frac{f(z)}{z} = 1 + az^p + \eta(z)z^p,$$

where $\eta(z) = o(1)$ as $z \to 0$.

Similarly

$$f'(z) = 1 + a(p+1)z^p + \beta(z)z^p,$$

where $\beta(z) = o(1)$ as $z \to 0$. Let us choose $\widetilde{\epsilon} > 0$ such that $|\eta(z)| < |a|/2$ and $|\beta(z)| < |a|/2$ for all z with $0 < |z| < \widetilde{\epsilon}$. Suppose first that $\mathbf{v} \in \mathbb{R}^+$, which implies by the definition of repelling vectors that $a > 0$. It follows that

$$\begin{aligned} \frac{|f(z)|}{|z|} &= |1 + az^p + \eta(z)z^p| \geq \mathrm{Re}(1 + az^p + \eta(z)z^p) \\ &= 1 + a\mathrm{Re}(z^p) + \mathrm{Re}(\eta(z)z^p) \geq 1 + a\mathrm{Re}(z^p) - |\eta(z)||z|^p \\ &> 1 + a\mathrm{Re}(z^p) - (a/2)|z|^p. \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \mathrm{Re}(f'(z)) &= 1 + a(p+1)\mathrm{Re}(z^p) + \mathrm{Re}(\beta(z)z^p) \\ &\geq 1 + a(p+1)\mathrm{Re}(z^p) - |\beta(z)||z|^p \\ &> 1 + a(p+1)\mathrm{Re}(z^p) - (a/2)|z|^p. \end{aligned} \tag{3.3}$$

for all z with $0 < |z| < \widetilde{\epsilon}$.

Note that

$$\mathrm{Re}(z^p) = \mathrm{Re}(|z|^p e^{ip\mathrm{Arg}(z)}) = |z|^p \cos(p\mathrm{Arg}(z)).$$

So if $|\mathrm{Arg}(z)| < \pi/(4p)$ then $\mathrm{Re}(z^p) > (\sqrt{2}/2)|z|^p$. Thus, by (3.2) and (3.3) we have that

$$\frac{|f(z)|}{|z|} > 1 + a(\sqrt{2}/2)|z|^p - (a/2)|z|^p = 1 + \left(\sqrt{2} - 1\right)(a/2)|z|^p > 1,$$

and that

$$\mathrm{Re}(f'(z)) > 1 + a(p+1)(\sqrt{2}/2)|z|^p - (a/2)|z|^p = 1 + \left((p+1)\sqrt{2} - 1\right)(a/2)|z|^p.$$

Note that $(p+1)\sqrt{2} - 1 > 0$. Hence, we have

$$|f'(z)| \geq \mathrm{Re}(f'(z)) > 1.$$

Now, if $\zeta \neq 0$ and $\mathbf{v} \notin \mathbb{R}^+$ then we conjugate f by $z \mapsto (z - \zeta)/\mathbf{v}$ to a map g such that $g(w) := (f(\mathbf{v}w + \zeta) - \zeta)/\mathbf{v}$. Since $f(\zeta) = \zeta$ and $f'(\zeta) = 1$ then g has a parabolic fixed point at zero with multiplier one. It is clear that g has a repelling vector that belongs to $\mathbb{R}^+$. Thus, there exists $\widetilde{\epsilon} > 0$ such that $|g(w)| > |w|$ and $|g'(w)| > 1$ for all w with $0 < |w| < \widetilde{\epsilon}$ and $|\mathrm{Arg}(w)| < \pi/(4p_\zeta)$. Hence, by choosing $\epsilon := \widetilde{\epsilon}\,|\mathbf{v}|$ we obtain

$$|f(z) - \zeta| > |z - \zeta|,$$

and

$$|f'(z)| > 1,$$

for all z with $0 < |z - \zeta| < \epsilon$ and $|\mathrm{Arg}\left((z - \zeta)/\mathbf{v}\right)| < \pi/(4p_\zeta)$, as claimed. □

3.2 Attracting and repelling petals

We are going to associate so called ”petals” to each attracting(repelling) vector. There seems to be no standard definition for the concept of a petal and different authors give different conditions [Mil06, page 11]. So we will define attracting and repelling petals as in [Mil06, Definition 10.6].

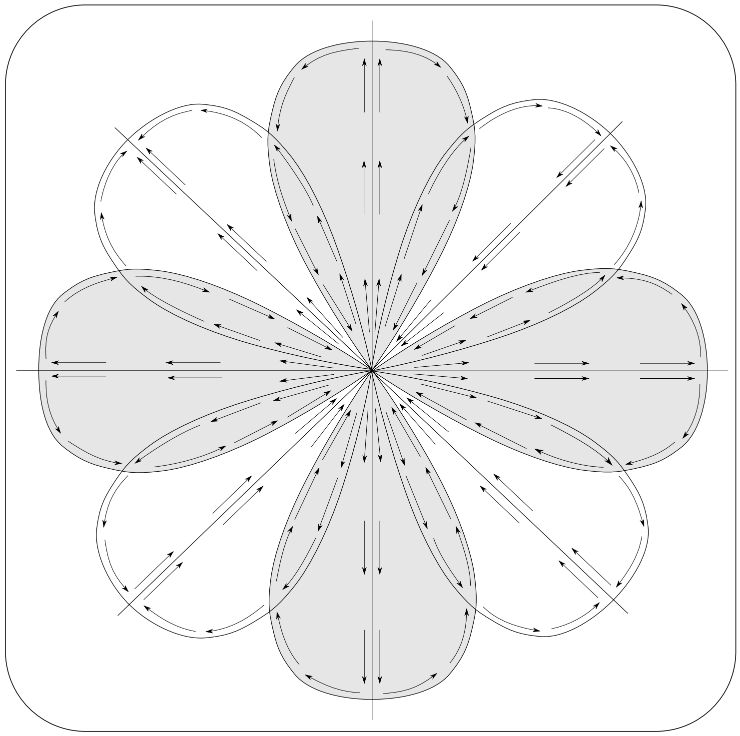

Figure 3.1: A parabolic point with four attracting(repelling) petals.

Definition 3.7 (Attracting and repelling petals)**.** *Let f be holomorphic and univalent on a neighbourhood $\mathcal{N}$ of $\zeta \in \mathbb{C}$. Suppose that ζ is a parabolic fixed point of f with multiplier one. Then a nonempty open set $\mathcal{P}_{\mathrm{A}} \subset \mathcal{N}$ is called an attracting petal for f at ζ if the following hold:*

(a) $f(\mathcal{P}_{\mathrm{A}}) \subset \mathcal{P}_{\mathrm{A}}$.

(b) *There exists an attracting vector $\mathbf{v}_a$ at ζ such that, if $z \in \mathcal{N}$, then the orbit of z converges to ζ from the direction $\mathbf{v}_a$ if and only if there exists $N \in \mathbb{N}$ such that $f^k(z) \in \mathcal{P}_{\mathrm{A}}$ for all $k \geq N$.*

Set $f(\mathcal{N}) = \widetilde{\mathcal{N}}$. Then an open set $\mathcal{P}_{\mathrm{R}} \subset \widetilde{\mathcal{N}}$ is a repelling petal for a repelling vector $\mathbf{v}_r$ if $\mathcal{P}_{\mathrm{R}}$ is an attracting petal for the map $f^{-1} \colon \widetilde{\mathcal{N}} \to \mathcal{N}$.

In [Mil06, Theorem10.7], the author proved the existence of bounded simply connected petals for every attracting and repelling vector at a parabolic fixed point $\zeta \in \mathbb{C}$. We will state this result below.

Theorem 3.8. *If ζ is a parabolic fixed point of multiplicity $p+1$, then within any neighbourhood of ζ there exist simply connected petals $\mathcal{P}_j$, where the subscript j ranges over the integers modulo $2p$ and where $\mathcal{P}_j$ is either repelling or attracting according to whether j is even or odd. Furthermore, these petals can be chosen so that the union*

$$\{\zeta\} \cup \mathcal{P}_0 \cup \ldots \cup \mathcal{P}_{2p-1}.$$

When $p > 1$, each $\mathcal{P}_j$ intersects each of its immediate neighbours in a simply connected region but is disjoint from the remaining $\mathcal{P}_k$.

In this section we will prove the existence of attracting and repelling petals with certain additional properties that are required for our results.
Before proving the existence of those attracting and repelling petals which we call *well-behaved petals* we will give their definitions.

Definition 3.9 (well-behaved attracting petals)**.** *Suppose that f has a parabolic fixed point at ζ with multiplier one and multiplicity $p_\zeta + 1$. Let $\mathcal{P}_\mathrm{A}$ be an attracting petal at ζ and $\mathbf{v}_\mathrm{A}$ be the attracting vector associated to it. Then $\mathcal{P}_\mathrm{A}$ is* well-behaved *if it has the following properties:*

(a) *$\mathcal{P}_\mathrm{A}$ makes an angle θ at ζ centered at $\mathbf{v}_\mathrm{A}$ where $0 < \theta < 2\pi/p_\zeta$.*

(b) *$\mathcal{P}_\mathrm{A}$ is simply connected.*

(c) *$f\left(\overline{\mathcal{P}}_\mathrm{A} \setminus \{\zeta\}\right) \subsetneq \mathcal{P}_\mathrm{A}$.*

Remark 3.10. *Suppose that $\zeta \in \mathrm{Par}(f)$ and $\mathcal{P}_\mathrm{A}$ is a well-behaved attracting petal at ζ with a vector $\mathbf{v}_\mathrm{A}$ associated to it. Let $0 < \alpha < 2\pi$ and define*

$$\Delta_\alpha := \{z\mathbf{v}_\mathrm{A} \in \mathbb{C} \colon |z| > 0, |\mathrm{Arg}(z - \zeta)| < \alpha/2\}.$$

Then we say $\mathcal{P}_{\mathrm{A}}$ makes an angle $0<\theta<2\pi$ at ζ, if for all $0<\varphi<\theta$ there exists $r=r(\varphi)>0$ such that

$$\Delta_{\theta-\varphi}\cap D(\zeta,r)\subset\mathcal{P}_{\mathrm{A}}\cap D(\zeta,r)\subset\Delta_{\theta+\varphi}\cap D(\zeta,r).$$

Definition 3.11 (well-behaved repelling petals)**.** *Let f be as in Definition 3.9. Let $\mathcal{P}_{\mathrm{R}}$ be a repelling petal at ζ and let g be a local inverse of f for which $\mathcal{P}_{\mathrm{R}}$ is an attracting petal. Then $\mathcal{P}_{\mathrm{R}}$ is well-behaved if it is a well-behaved attracting petal for g.*

For $R>0$ and $0<\alpha<2\pi$, we will define new sets $S_{R,\alpha}$ and $S_{-R,\alpha}$ as follows:

$$S_{R,\alpha}:=\{w\in\mathbb{C}\colon |\mathrm{Arg}(w-R)|<\alpha/2\},\tag{3.4}$$

and

$$S_{-R,\alpha}:=\{w\in\mathbb{C}\colon |\mathrm{Arg}(w+R)|>\alpha/2\}.\tag{3.5}$$

Let

$$\kappa\colon\widehat{\mathbb{C}}\to\widehat{\mathbb{C}},\quad\text{with}\quad\kappa(z):=\frac{-1}{paz^p}.\tag{3.6}$$

In the proof of Theorem 10.7 in [Mil06] the author proved that the preimages of $S_{R,3\pi/2}$ and $S_{-R,3\pi/2}$ under κ for sufficiently large $R>0$ are attracting and repelling petals with useful properties. In the next two results we will prove that the preimages of $S_{R,\alpha}$ and $S_{-R,\alpha}$ under κ are attracting and repelling petals, respectively, for sufficiently large $R>0$ and $0<\alpha<2\pi$. We will also show that these petals have additional properties that are required for our results. In the beginning of our proof we will use similar arguments to those used in the proof of [Mil06, Lemma10.1] and [Mil06, Theorem 7.10], but we will discuss the steps of the argument with more details.

Suppose that f has a parabolic fixed point at zero with multiplier one and multiplicity $p+1$, i.e. f can be written as in (3.1). Then f has p attracting and p repelling vectors at zero. Let $\mathbf{v}_{\mathrm{A}}$ be an attracting vector and $\mathbf{v}_{\mathrm{R}}$ be a repelling vector at zero. For $0 < \alpha < 2\pi$, we will define the open sectors Δ_{Att} and Δ_{Rep} as follows.

$$\Delta_{\mathrm{Att}} := \{re^{i\theta}\mathbf{v}_{\mathrm{A}} \colon r > 0 \text{ and } |\theta| < \alpha/(2p)\}, \tag{3.7}$$

and

$$\Delta_{\mathrm{Rep}} := \{re^{i\theta}\mathbf{v}_{\mathrm{R}} \colon r > 0 \text{ and } |\theta| < \alpha/(2p)\}. \tag{3.8}$$

Let $z = re^{i\theta}\mathbf{v}_{\mathrm{A}} \in \Delta_{\mathrm{Att}}$ then by the definition of attracting vectors at zero

$$\kappa(z) = \frac{-1}{par^p e^{ip\theta}\mathbf{v}_{\mathrm{A}}^p} = \frac{1}{r^p e^{ip\theta}} = \frac{1}{r^p} e^{-ip\theta}.$$

Since $\dfrac{-\alpha}{2p} < \theta < \dfrac{\alpha}{2p}$ then

$$\frac{-\alpha}{2} < -p\theta < \frac{\alpha}{2}.$$

This implies that κ maps Δ_{Att} to the set

$$S_\alpha := \{w \in \mathbb{C} \colon |\mathrm{Arg}(w)| < \alpha/2\}. \tag{3.9}$$

Similarly, if $z = re^{i\theta}\mathbf{v}_{\mathrm{R}} \in \Delta_{\mathrm{Rep}}$ then by the definition of repelling vectors at zero

$$\kappa(z) = \frac{-1}{par^p e^{ip\theta}\mathbf{v}_{\mathrm{R}}^p} = \frac{-1}{r^p e^{ip\theta}} = \frac{1}{r^p}\left(-e^{-ip\theta}\right) = \frac{1}{r^p} e^{i(\pi - p\theta)}.$$

Again since $\dfrac{-\alpha}{2p} < \theta < \dfrac{\alpha}{2p}$ then

$$\pi - \frac{\alpha}{2} < \pi - p\theta < \pi + \frac{\alpha}{2}.$$

Thus, κ maps Δ_{Rep} to the set

$$\widetilde{S}_\alpha := \{w \in \mathbb{C} \colon |\mathrm{Arg}(w)| > \alpha/2\}. \tag{3.10}$$

Observe that κ is bijective on the open sectors Δ_{Att} and Δ_{Rep}. Hence, there are uniquely defined inverse branches ψ_{A} and ψ_{R} of κ such that

$$\psi_{\text{A}} \colon S_\alpha \to \mathbb{C}, \qquad \text{with} \qquad \psi_{\text{A}}(w) = \left(\frac{-1}{paw}\right)^{1/p} = \frac{\mathbf{v}_{\text{A}}}{w^{1/p}}, \tag{3.11}$$

where the branch of $w^{1/p}$ is the one defined on $\mathbb{C}\backslash(-\infty, 0]$ and maps 1 to 1, and

$$\psi_{\text{R}} \colon \widetilde{S}_\alpha \to \mathbb{C}, \qquad \text{with} \qquad \psi_{\text{R}}(w) = \left(\frac{-1}{paw}\right)^{1/p} = \mathbf{v}_{\text{R}} \left(\frac{-1}{w}\right)^{1/p}, \tag{3.12}$$

where the branch of $(-1/w)^{1/p}$ is the one defined on $\mathbb{C}\backslash[0, \infty)$ and maps -1 to 1.

Observe that $S_{R,\alpha} \subset S_\alpha$ and $S_{-R,\alpha} \subset \widetilde{S}_\alpha$ for all $R > 0$.

Lemma 3.12 (Existence of well-behaved attracting petals)**.** *Suppose that f is an entire function that has a parabolic fixed point at $\zeta \in \mathbb{C}$ with multiplicity $p+1$ and multiplier one. Let ψ_{A} be as in (3.11) and ϕ be the translation $z \mapsto z + \zeta$. Suppose that $0 < \alpha < 2\pi$ and $S_{R,\alpha}$ is defined as in (3.4). Then there exists $\widetilde{R} > 0$ such that for all $R \geq \widetilde{R}$ the set $\mathcal{P}_{\text{A}}(R) := \phi\left(\psi_{\text{A}}(S_{R,\alpha})\right)$ is a well-behaved attracting petal for f with an angle α/p at ζ.*

Moreover, for all sufficiently large $R > 0$ there exists $r(R) > 0$ such that $\mathcal{P}_{\text{A}}(R) \subset D(\zeta, r(R))$ and $r(R) \to 0$ as $R \to \infty$.

Proof. Let us first assume that $\zeta = 0$. So f can be written as in (3.1). Let Δ_{Att} be defined as in (3.7) where $\mathbf{v}_{\text{A}}$ is an attracting vector at ζ. Since $S_{R,\alpha} \subset S_\alpha$ then $\psi_{\text{A}}(S_{R,\alpha}) \subset \psi_{\text{A}}(S_\alpha) = \Delta_{\text{Att}}$. We will study the behaviour of the function f near zero in the sector Δ_{Att}, and to do so we conjugate f to a map F_{A} as follows

$$F_{\text{A}} \colon S_{R,\alpha} \to \widehat{\mathbb{C}}, \qquad F_{\text{A}}(w) := \kappa \circ f \circ \psi_{\text{A}}(w). \tag{3.13}$$

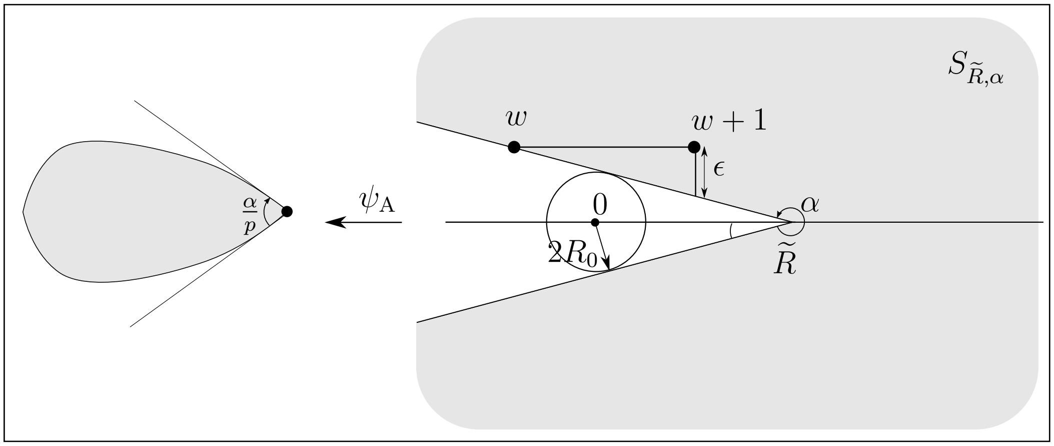

Figure 3.2: Illustration of Lemma 3.12 for $\pi < \alpha < 2\pi$. Here $\epsilon = \tan(\pi - \alpha/2)$ and $\widetilde{R} = \dfrac{2R_0}{\sin(\alpha/2)}$.

where κ is given in (3.6).

It follows from equations (3.1) and (3.11) that

$$f\left(\psi_{\mathrm{A}}(w)\right) = \left(\frac{-1}{paw}\right)^{1/p}\left(1 - \frac{1}{pw} + \dots\right).$$

Then, by (3.6) we have

$$\begin{aligned}\kappa\left(f\left(\psi_{\mathrm{A}}(w)\right)\right) =& \frac{-1}{pa\left(\frac{-1}{paw}\right)\left(1 - \frac{1}{pw} + \dots\right)^{p}} = \frac{w}{\left(1 - \frac{1}{pw} + \dots\right)^{p}}\\ =& \frac{w}{(1 - (1/w) + \dots)}.\end{aligned}$$

Hence, the function F_{A} can be written in the form

$$F_{\mathrm{A}}(w) = w + 1 + O(1/w), \quad \text{as} \quad w \to \infty.$$

So there are constants C and $R' > 0$ such that

$$|F_{\mathrm{A}}(w) - w - 1| < \frac{C}{|w|}, \quad \text{for } \ |w| > R'. \tag{3.14}$$

Set

$$R_0 := \begin{cases} \max\{R', 1 + 32C\}, & \text{if } \ 0 < \alpha \leq \pi, \\ \max\{R', 1 + 32C/\sin(\alpha/2)\}, & \text{if } \ \pi < \alpha < 2\pi. \end{cases} \tag{3.15}$$

We consider first the case $0 < \alpha \leq \pi$. We will choose $\widetilde{R} = 2R_0$. In this case if $R \geq \widetilde{R}$ then the set $S_{R,\alpha}$ is contained in the right half plane $\mathbb{H}_r(R)$. So if $w \in \overline{S}_{R,\alpha}$ then $\mathrm{Re}(w) \geq R \geq 2R_0 > R'$. By (3.15) we have $R_0 > 2C$. Then by (3.14), we obtain

$$|F_{\mathrm{A}}(w) - w - 1| < \frac{C}{R_0} < \frac{1}{2}.$$

It follows that

$$\mathrm{Re}(F_{\mathrm{A}}(w)) \geq \mathrm{Re}(w) + 1/2 \geq R + 1/2,$$

and hence $F_{\mathrm{A}}(w) \in S_{R,\alpha}$. This implies that $F_{\mathrm{A}}(\overline{S}_{R,\alpha}) \subsetneq S_{R,\alpha}$ for $0 < \alpha \leq \pi$.

Now, we consider the case $\pi < \alpha < 2\pi$. Let us define $\widetilde{R} := 2R_0/\sin(\alpha/2)$, see Figure 3.2. Since $0 < \sin(\alpha/2) < 1$ for $\pi < \alpha < 2\pi$ then $\widetilde{R} > 2R_0$. So if $R \geq \widetilde{R}$ and $w \in \overline{S}_{R,\alpha}$ then $|w| \geq \ 2R_0 > R'$. Then it follows from (3.14) and (3.15) that

$$|F_{\mathrm{A}}(w) - w - 1| < \frac{C}{R_0} < \sin(\alpha/2) \leq \tan(\pi - \alpha/2),$$

which implies that $F_{\mathrm{A}}(w) \in S_{R,\alpha}$, see Figure 3.2. We can deduce that

$$F_{\mathrm{A}}(\overline{S}_{R,\alpha}) \subsetneq S_{R,\alpha} \quad \text{for} \quad \pi < \alpha < 2\pi.$$

It follows from the two cases above that

$$F_{\mathrm{A}}(\overline{S}_{R,\alpha}) \subsetneq S_{R,\alpha}, \quad \text{for} \;\; R \geq \widetilde{R} \;\; \text{and} \;\; 0 < \alpha < 2\pi. \tag{3.16}$$

We will now prove that $\psi_{\mathrm{A}}(S_{R,\alpha})$ is an attracting petal. First, note that it is an open set because ψ_{A} is holomorphic. Since $F_{\mathrm{A}}(S_{R,\alpha}) \subsetneq S_{R,\alpha}$ then by definition $f(\psi_{\mathrm{A}}(S_{R,\alpha})) \subset \psi_{\mathrm{A}}(S_{R,\alpha})$.

Let $R \geq \widetilde{R}$ and $w \in S_{R,\alpha}$. By induction we claim that for all $n \geq 1$,

$$|F_{\mathrm{A}}^{j}(w) - F_{\mathrm{A}}^{j-1}(w) - 1| < \frac{C}{|F_{\mathrm{A}}^{j-1}(w)|}, \quad 1 \leq j \leq n, \tag{3.17}$$

$$|F_{\mathrm{A}}^{n}(w) - w - n| < \sum_{j=0}^{n-1} \frac{C}{|F_{\mathrm{A}}^{j}(w)|}, \tag{3.18}$$

$$|F_{\mathrm{A}}^{n}(w) - w - n| < \sum_{j=0}^{n-1} \frac{2C}{|w+j|} < \sum_{j=0}^{n-1} \frac{8C}{R_0 + j} < 8C \log\left(1 + \frac{n}{R_0 - 1}\right), \tag{3.19}$$

$$|F_{\mathrm{A}}^{j}(w)| > \frac{|w+j|}{2} > R_0, \quad 1 \leq j \leq n. \tag{3.20}$$

It is clear that the inequalities (3.17), (3.18) and the first inequality in (3.19) hold for $n = 1$. The second inequalities in (3.19) and (3.20) hold for $n = 1$ because $|w + n| > 2R_0$ for all $w \in S_{R,\alpha}$ and $n \geq 0$. Note that $R_0 > 1$ by (3.15). The last inequality in (3.19) holds for $n = 1$ because $\log x \geq 1 - 1/x$ for all $x > 0$. Since $|w| > 2R_0$ then it follows again from (3.14) and (3.15) that

$$|F_{\mathrm{A}}(w) - w - 1| < 1.$$

Thus, we have

$$|F_{\mathrm{A}}(w)| > |w+1|/2 + (|w+1|/2 - 1) > |w+1|/2 + (R_0 - 1) > |w+1|/2.$$

So the first inequality in (3.20) holds for $n = 1$.

Suppose now that the inequalities from (3.17) to (3.20) hold for $j = n - 1$. Then by (3.20) for $n - 1$ we have

$$|F_{\mathrm{A}}^{n-1}(w)| > \frac{|w + (n-1)|}{2} > R_0 \geq R'.$$

So (3.17) holds for $j = n$ by replacing w in (3.14) by $F_{\mathrm{A}}^{n-1}(w)$. Then adding (3.17) for $1 \leq j \leq n$ we obtain (3.18). By using (3.20) for $1 \leq j \leq n-1$ and (3.18) we obtain the first inequality in (3.19).

Now let n_0 be such that $\mathrm{Re}(w+n_0)$ is minimal among $w+n$ with $\mathrm{Re}(w+n) > 0$. To prove the second inequality of (3.19) we will first show that

$$|w + j| \geq \frac{|w + n_0| + (j - n_0)}{2}, \quad \text{for} \quad j \geq n_0, \tag{3.21}$$

and

$$|w + j| \geq \frac{|w + n_0 - 1| + (n_0 - 1 - j)}{2}, \quad \text{for} \quad 0 < j < n_0, \tag{3.22}$$

To prove (3.21) we will consider the following two cases. If $0 \leq j - n_0 \leq |w + n_0|$, then we have

$$|w + j| \geq |w + n_0| = \frac{|w + n_0|}{2} + \frac{|w + n_0|}{2} \geq \frac{|w + n_0| + (j - n_0)}{2}.$$

If $j - n_0 > |w + n_0|$, then we have

$$|w + j| \geq \mathrm{Re}(w + j) = \mathrm{Re}(w + n_0) + (j - n_0) > j - n_0 > \frac{|w + n_0| + (j - n_0)}{2}.$$

The inequality (3.22) is proved similarly by considering separately the cases $0 \leq n_0 - 1 - j \leq |w + n_0 - 1|$, and $n_0 - 1 - j > |w + n_0 - 1|$.

Recall that $|w+n| > 2R_0$ for all $n \geq 0$. Then if $n_0 = 0$ (always true if $\pi < \alpha < 2\pi$), then it follows from (3.21) that

$$\sum_{j=0}^{n-1} \frac{1}{|w+j|} \leq \sum_{j=0}^{n-1} \frac{1}{|w|+j} < \sum_{j=0}^{n-1} \frac{2}{R_0+j}.$$

If $n_0 > 0$ (which is possible for $0 < \alpha \leq \pi$), then it follows from (3.21) and (3.22) that

$$\begin{aligned}\sum_{j=0}^{n-1} \frac{1}{|w+j|} &= \sum_{j=0}^{n_0-1} \frac{1}{|w+j|} + \sum_{j=n_0}^{n-1} \frac{1}{|w+j|} \\ &\leq \sum_{j=0}^{n_0-1} \frac{2}{R_0+(n_0-1-j)} + \sum_{j=n_0}^{n-1} \frac{2}{R_0+(j-n_0)} \\ &= \sum_{j=0}^{n_0-1} \frac{2}{R_0+j} + \sum_{j=0}^{n-1-n_0} \frac{2}{R_0+j}.\end{aligned}$$

Since $n \geq n_0$ and $n > n - n_0$, then the second inequality in (3.19) holds for $j = n$. The third inequality in (3.19) holds by the Integral Test applied to the function $1/(R_0 - 1 + x)$ on the interval $0 \leq x \leq n$.

We will show now that the first inequality in (3.20) holds for $j = n$. Recall that $0 < \sin(\alpha/2) < 1$ for $\pi < \alpha < 2\pi$. So it follows from (3.15) that $R_0 > 1 + 32C$. So if $|w+n| > n/2$ then since $\log(1+x) \leq x$ for all $x \geq 0$ we have

$$8C \log\left(1 + \frac{n}{R_0 - 1}\right) < 8C \log\left(1 + \frac{n}{32C}\right) \leq \frac{n}{4} < \frac{|w+n|}{2}.$$

Thus by (3.19), we obtain

$$|F_{\mathrm{A}}^n(w)| > |w+n| - \frac{|w+n|}{2} = \frac{|w+n|}{2} > R_0.$$

If $|w+n| \le n/2$ then $\mathrm{Re}(w) \le -n/2$, and hence $\pi < \alpha < 2\pi$. Since $w \in S_{R,\alpha}$ then

$$\begin{aligned}|w+n| \ge \mathrm{Im}(w) &\ge \tan\left(\pi - (\alpha/2)\right)\left(R + |\mathrm{Re}(w)|\right) \\ &> 2R_0 + \sin\left(\alpha/2\right)|\mathrm{Re}(w)| > \sin\left(\alpha/2\right)|\mathrm{Re}(w)|.\end{aligned}$$

Since $R_0 \ge 1+32C/\sin(\alpha/2)$ and $2|\mathrm{Re}(w)| \ge n$ then again since $\log(1+x) \le x$ for all $x \ge 0$, we have

$$\begin{aligned}8C\log\left(1+\frac{n}{R_0-1}\right) &\le 8C\log\left(1+\frac{2|\mathrm{Re}(w)|}{R_0-1}\right) \\ &\le 8C\log\left(1+\frac{2\sin(\alpha/2)|\mathrm{Re}(w)|}{32C}\right) \\ &\le \frac{\sin(\alpha/2)|\mathrm{Re}(w)|}{2} < \frac{|w+n|}{2}.\end{aligned}$$

Hence, the inequality (3.20) holds for $j = n$.

Thus, we have

$$\mathrm{Im}\left(F_{\mathrm{A}}^{n}(w)\right) < \mathrm{Im}(w) - 8C\log\left(1+n/(R_0-1)\right),$$

and

$$\mathrm{Re}\left(F_{\mathrm{A}}^{n}(w)\right) > \mathrm{Re}(w) + n - 8C\log\left(1+n/(R_0-1)\right).$$

It follows that $\mathrm{Re}\left(F_{\mathrm{A}}^{n}(w)\right) \to \infty$ as $n \to \infty$, and

$$\frac{\mathrm{Im}\left(F_{\mathrm{A}}^{n}(w)\right)}{\mathrm{Re}\left(F_{\mathrm{A}}^{n}(w)\right)} \to 0, \quad \text{as} \quad n \to \infty.$$

This implies that $\mathrm{Arg}\left(F_{\mathrm{A}}^{n}(w)\right) \to 0$ as $n \to \infty$.

Hence $\psi_{\mathrm{A}}\left(F_{\mathrm{A}}^{n}(\kappa(z))\right) \to 0$ as $n \to \infty$ for $\kappa(z) \in S_{R,\alpha}$. This means that $f^n(z) \to 0$ as $n \to \infty$ for $z \in \psi_{\mathrm{A}}\left(S_{R,\alpha}\right)$. Since $\mathrm{Arg}\left(F_{\mathrm{A}}^{n}(w)\right) \to 0$ as

$n \to \infty$ then $\mathrm{Arg}\left(\psi_{\mathrm{A}}\left(F_{\mathrm{A}}^{n}(\kappa(z))\right)\right) \to \mathbf{v}_{\mathrm{A}}$ as $n \to \infty$. This means that $\mathrm{Arg}\left(f^{n}(z)\right) \to \mathbf{v}_{\mathrm{A}}$ as $n \to \infty$ for $z \in \psi_{\mathrm{A}}\left(S_{R,\alpha}\right)$.

Let $\mathcal{N}$ be a neighbourhood of zero such that $\psi_{\mathrm{A}}\left(S_{R,\alpha}\right) \subset \mathcal{N}$ and f is holomorphic and univalent on $\mathcal{N}$. If $z \in \mathcal{N}$ and $f^{n}(z) \to 0$ from the direction $\mathbf{v}_{\mathrm{A}}$ then this means that $\mathrm{Arg}\left(f^{n}(z)\right) \to \mathrm{Arg}\left(\mathbf{v}_{\mathrm{A}}\right)$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $f^{n}(z) \in \psi_{\mathrm{A}}\left(S_{R,\alpha}\right)$ for all $n > N$. This proves that the set $\psi_{\mathrm{A}}\left(S_{R,\alpha}\right)$ is an attracting petal.

Since ψ_{A} is a homeomorphism from S_{α} to Δ_{Att} and $S_{\alpha,R} \subset S_{\alpha}$, then the set $\psi_{\mathrm{A}}(S_{R,\alpha})$ is contained in the sector Δ_{Att}. It is clear geometrically that is for all $0 < \varphi < \alpha/p$ there exists $r = r(\varphi) > 0$ such that

$$\Delta_{\alpha/p-\varphi} \cap D(0,r) \subset \psi_{\mathrm{A}}(S_{R,\alpha}) \cap D(0,r) \subset \Delta_{\alpha/p} \cap D(0,r).$$

Hence, $\psi_{\mathrm{A}}(S_{R,\alpha})$ makes an angle α/p at zero by Remark 3.10. Moreover, since $S_{R,\alpha}$ is simply connected and again ψ_{A} is a homeomorphism then the set $\psi_{\mathrm{A}}(S_{R,\alpha})$ is also simply connected.

Since $\overline{S}_{R,\alpha}$ is the closure of $S_{R,\alpha}$ in $\mathbb{C}$ then (3.16) implies that

$$f\left(\overline{\psi_{\mathrm{A}}(S_{R,\alpha})}\backslash\{0\}\right) \subsetneq \psi_{\mathrm{A}}\left(S_{R,\alpha}\right).$$

This proves that $\psi_{\mathrm{A}}(S_{R,\alpha})$ is a well-behaved attracting petal at zero.

Observe that

$$S_{R,\alpha} \subset \{w \in \mathbb{C} \colon |w| \geq R\}, \quad \text{for} \quad 0 < \alpha \leq \pi.$$

and

$$S_{R,\alpha} \subset \{w \in \mathbb{C} \colon |w| > R\sin((2\pi-\alpha)/2)\}, \quad \text{for} \quad \pi < \alpha < 2\pi.$$

Hence, by definition

$$\psi_{\mathrm{A}}(S_{R,\alpha}) \subset D\left(0, (p|a|/R)^{1/p}\right), \quad \text{for} \quad 0 < \alpha \le \pi$$

and

$$\psi_{\mathrm{A}}(S_{R,\alpha}) \subset D\left(0, (p|a|/R\sin\left((2\pi-\alpha)/2\right))^{1/p}\right), \quad \text{for} \quad \pi < \alpha < 2\pi.$$

Define

$$\widetilde{r}(R) := (p|a|)^{1/p}\,(1/R)^{1/p}, \quad \text{for} \quad 0 < \alpha \le \pi,$$

and

$$\widetilde{r}(R) := (p|a|/\sin((2\pi-\alpha)/2))^{1/p}\,(1/R)^{1/p}, \quad \text{for} \quad \pi < \alpha < 2\pi.$$

then we have $\widetilde{r}(R) \to 0$ as $R \to \infty$. This completes the proof in the case that the parabolic fixed point is at zero.

Now, if $\zeta \neq 0$ we conjugate f to a map $g := \phi \circ f \circ \phi^{-1}$. Then we prove that $\psi_{\mathrm{A}}(S_{R,\alpha})$ is a well-behaved attracting petal for g at zero with the property that there exits $\widetilde{r}(R) > 0$ such that $\mathcal{P}_{\mathrm{A}}(R) \subset D(\zeta, \widetilde{r}(R))$ and $\widetilde{r}(R) \to 0$ as $R \to \infty$. Since ϕ is a homeomorphism then the set $\mathcal{P}_A(R) = \phi\left(\psi_{\mathrm{A}}(S_{R,\alpha})\right)$ is a well-behaved attracting petal for f at ζ with the property that there exits $r(R) > 0$ such that $\mathcal{P}_{\mathrm{A}}(R) \subset D(\zeta, r(R))$ and $r(R) \to 0$ as $R \to \infty$. $\square$

Lemma 3.13 (Existence of well-behaved repelling petals)**.** *Let f and ϕ be as in Lemma 3.12. Let ψ_{R} be the function defined in (3.12). Suppose that $0 < \alpha < 2\pi$ and $S_{-M,\alpha}$ is defined as in (3.5) for $M > 0$. Then there exists $\widetilde{M} > 0$ such that for all $M \ge \widetilde{M}$ the set $\mathcal{P}_{\mathrm{R}}(M) := \phi\left(\psi_{\mathrm{R}}(S_{-M,\alpha})\right)$ is a well-behaved repelling petal for f with an angle α/p at ζ.*

Moreover, for all sufficiently large $M > 0$ there exists $r(M) > 0$ such that $\mathcal{P}_{\mathrm{R}}(M) \subset D(\zeta, r(M))$ and $r(M) \to 0$ as $M \to \infty$.

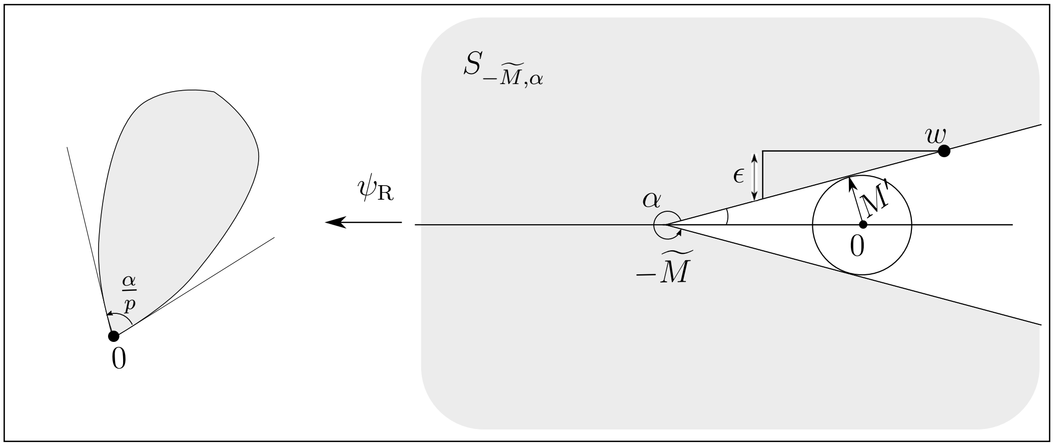

Figure 3.3: Illustration of Lemma 3.13 for $\pi < \alpha < 2\pi$. Here $\epsilon = \tan(\pi - \alpha/2)$ and $\widetilde{M} = \dfrac{M'}{\sin(\alpha/2)}$.

Proof. By definition the set $\mathcal{P}_{\mathrm{R}}(M)$ is a repelling petal for f if it is an attracting petal for a local inverse of f. As we are going to prove that $\mathcal{P}_{\mathrm{R}}(M)$ is an attracting petal for a local inverse g of f, we will omit some details already discussed in the proof of Lemma 3.12.

Let us first assume that $\zeta = 0$. Let Δ_{Rep} be defined as in (3.8) where $\mathbf{v}_{\mathrm{R}}$ is a repelling vector of f. Since $S_{-M,\alpha} \subset \widetilde{S}_\alpha$ then $\psi_{\mathrm{R}}(S_{-M,\alpha}) \subset \psi_{\mathrm{R}}\left(\widetilde{S}_\alpha\right) = \Delta_{\mathrm{Rep}}$. So we will study the behaviour of f near the parabolic fixed point zero in the sector Δ_{Rep}. We conjugate f to a map F_{R} as follows.

$$F_{\mathrm{R}}\colon S_{-M,\alpha} \to \mathbb{C}, \qquad F_{\mathrm{R}}(w) := \kappa \circ f \circ \psi_{\mathrm{R}}(w). \tag{3.23}$$

where κ is given in (3.6).

By the same calculations as in the beginning of the proof of Lemma 3.12, we have

$$F_{\mathrm{R}}(w) = w + 1 + o(1), \qquad \text{as} \qquad |w| \to \infty. \tag{3.24}$$

Suppose that G is a local inverse of F_{R}. Then G can be written as

$$G(w) = w - 1 + o(1), \qquad \text{as} \qquad |w| \to \infty.$$

Now, we will consider two cases. For the case $\pi < \alpha < 2\pi$, define $\epsilon := \tan(\pi - \alpha/2)$. Then we choose $M' > 0$ large such that

$$|G(w) - w + 1| < \min\{\frac{1}{2}, \epsilon\}, \qquad \text{for} \quad |w| > M'.$$

Then we define $\widetilde{M} := M'/\sin(\alpha/2)$, see Figure 3.3. Note that $\widetilde{M} > M'$. So, if $M \geq \widetilde{M}$ and $w \in \partial S_{-M,\alpha}$ then $|w| > M'$ and by the choice of ϵ we have

$$G\left(\partial S_{-M,\alpha}\right) \subset S_{-M,\alpha}.$$

which implies that

$$G(\overline{S}_{-M,\alpha}) \subsetneq S_{-M,\alpha}.$$

For the second case when $0 < \alpha \leq \pi$, we will choose $\widetilde{M} > 0$ such that

$$|G(w) - w + 1| \leq \frac{1}{2}, \qquad \text{for} \quad |w| \geq \widetilde{M}.$$

Then, we have

$$\mathrm{Re}\left(G(w)\right) \leq \mathrm{Re}\left(w\right) - 1/2, \qquad \text{for} \quad |w| \geq \widetilde{M}.$$

In this case the set $S_{-M,\alpha}$ is contained in the left half plane $\mathbb{H}_l(M)$. Thus, if $M \geq \widetilde{M}$ and $w \in \overline{S}_{-M,\alpha}$ then $|w| \geq \widetilde{M}$ and

$$\mathrm{Re}(G(w)) \leq -M - 1/2,$$

and hence $G(w) \in S_{-M,\alpha}$. This implies that the function G satisfies $G(\overline{S}_{-M,\alpha}) \subsetneq S_{-M,\alpha}$

for $0 < \alpha \leq \pi$. Hence, it follows from the two cases above that

$$G(\overline{S}_{-M,\alpha}) \subsetneq S_{-M,\alpha}, \tag{3.25}$$

for all $M \geq \widetilde{M}$ and $0 < \alpha < 2\pi$.

By a similar discussion as in the proof of Lemma 3.12 we can prove that $\psi_{\mathrm{R}}(S_{-M,\alpha})$ is an attracting petal for a local inverse $g := \psi_{\mathrm{R}} \circ G \circ \kappa$ of the function f at zero. Hence, it is a repelling petal for f at zero.

Now, we will show that the set $\psi_{\mathrm{R}}(S_{-M,\alpha})$ is a well-behaved attracting petal for the local inverse g at zero. Since ψ_{R} is a homeomorphism from $\widetilde{S}_\alpha$ to Δ_{Rep} and $S_{-M,R} \subset \widetilde{S}_\alpha$, then the set $\psi_{\mathrm{R}}(S_{-M,\alpha})$ is contained in the sector Δ_{Rep}. Observe that ψ_{R} maps ∞ to zero. Thus, the boundary of $\psi_{\mathrm{R}}(S_{-M,\alpha})$ tends to the boundary of $\psi_{\mathrm{A}}(\widetilde{S}_\alpha)$ in a neighbourhood of zero. Hence, $\psi_{\mathrm{R}}(S_{-M,\alpha})$ makes an angle α/p at zero. Moreover, the set $\psi_{\mathrm{R}}(S_{-M,\alpha})$ is simply connected because the set $S_{-M,\alpha}$ is simply connected and ψ_{R} is a homeomorphism. Since $\overline{S}_{-M,\alpha}$ is the closure of $S_{-M,\alpha}$ in $\mathbb{C}$ then it follows from (3.25) that

$$g\left(\overline{\psi_{\mathrm{R}}\left(S_{-M,\alpha}\right)}\backslash\{0\}\right) \subsetneq \psi_{\mathrm{R}}(S_{-M,\alpha}).$$

This proves that $\psi_{\mathrm{R}}(S_{-M,\alpha})$ is a well-behaved attracting petal for g at zero. From here, the steps are the same as in the proof of Lemma 3.12. □

Remark 3.14. *By definition, the set $\kappa^{-1}\left(S_{R,\alpha}\right)$ has p components (one for each branch of the inverse) each of them makes an angle α/p at zero. These components are arranged symmetrically around zero. Likewise, the set $\kappa^{-1}\left(S_{-M,\alpha}\right)$ has p components each of them makes an angle α/p at zero, and they are arranged symmetrically around zero.*

We will now define the *parabolic basin* of a parabolic point. Suppose first that ζ is a parabolic fixed point with multiplier one. Then there is a *parabolic basin* $\mathcal{A}(\mathbf{v})$ associated to each attracting vector $\mathbf{v}$ at ζ, defined to be the set consisting of all points $z \in \mathbb{C}$ whose orbits $(f^n(z))_{n\geq 0}$ converge to ζ from the direction $\mathbf{v}$. The *immediate parabolic basin* $\mathcal{A}_0(\mathbf{v})$ is the unique connected component of $\mathcal{A}(\mathbf{v})$ that is mapped into itself under f. Each immediate parabolic basin has a unique parabolic point on its boundary. Parabolic basins associated to different attracting vectors at ζ are clearly pairwise disjoint. By definition, each attracting petal $\mathcal{P}_{\mathrm{A}}$ associated to an attracting vector $\mathbf{v}$ is contained in the immediate parabolic basin $\mathcal{A}_0(\mathbf{v})$ which consists of all points whose orbits eventually land in $\mathcal{P}_{\mathrm{A}}$, and hence converge to ζ from the direction $\mathbf{v}$.

The following results are for parabolic periodic points and their basins.

Lemma 3.15. [Mil06, Lemma 4.7] *Every parabolic periodic point of f belongs to the Julia set $\mathcal{J}(f)$.*

Lemma 3.16. [Mil06, Lemma 10.5] *Each parabolic basin is contained in the Fatou set $\mathcal{F}(f)$, but the boundary of each parabolic basin is contained in the Julia set $\mathcal{J}(f)$.*

3.3 Fatou coordinates

The following result was proved by Leau and Fatou [Mil06, Theorem 10.9]. It gives us some useful tools to study the dynamics of a function near a parabolic point and, in particular, in the petals.

Theorem 3.17 (Fatou coordinates)**.** *Let f be as in (3.1) and let $\mathcal{P}$ be any attracting or repelling petal at zero. Then there exists one and, up to a composition with a translation, only one conformal map $\phi : \mathcal{P} \to \mathbb{C}$ that satisfies*

$$\phi(f(z)) = \phi(z) + 1, \tag{3.26}$$

for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.

The map ϕ is often referred to as the *Fatou coordinate* in $\mathcal{P}$. The image $\phi(\mathcal{P}) \subset \mathbb{C}$ will contain a right half plane if $\mathcal{P}$ is an attracting petal, or a left half plane if $\mathcal{P}$ is a repelling petal.

Now, let ϕ be the Fatou coordinate on an attracting or a repelling petal $\mathcal{P}$ at zero. Suppose that κ given by (3.6) is defined on $\mathcal{P}$ and $\psi\colon \kappa(\mathcal{P}) \to \mathbb{C}$ is a branch of its inverse. We define the *Fatou coordinate at infinity* as follows.

$$\Phi\colon \kappa\left(\mathcal{P}\right) \to \mathbb{C}, \qquad \Phi(w) := \phi \circ \psi(w). \tag{3.27}$$

Note that Φ is conformal on $\kappa(\mathcal{P})$ by definition. Let $F\colon \kappa(\mathcal{P}) \to \mathbb{C}$ be defined as $F := \phi \circ f \circ \psi$, then it follows from (3.26) that

$$\Phi\left(F\left(w\right)\right) = \phi \circ \psi \circ F(w) = \phi \circ f \circ \psi(w) = \phi\left(\psi(w)\right) + 1 = \Phi(w) + 1, \tag{3.28}$$

for all $w \in \kappa(\mathcal{P} \cap f^{-1}(\mathcal{P}))$.

Remark. If $\mathcal{P}$ is an attracting petal then by definition $\mathcal{P} \cap f^{-1}(\mathcal{P}) = \mathcal{P}$ as f is univalent on $\mathcal{P}$. Similarly, if $\mathcal{P}$ is a repelling petal then $\mathcal{P} \cap f^{-1}(\mathcal{P}) = f^{-1}(\mathcal{P})$.

The following Lemma is a simple rescaling of the Koebe Distortion Theorem on the disc $D(0,2)$. We are going to use it in the proof of Lemma 3.19 below.

Lemma 3.18. *Let g be a conformal map defined on the disc $D(0,2)$, such that $g(0) = 0$ and $g'(0) = 1$. Then*

$$\frac{4|z|}{(2+|z|)^2} \leq |g(z)| \leq \frac{4|z|}{(2-|z|)^2}.$$

Proof. Note that the disc $D(0,2)$ is mapped conformally by $z \mapsto \dfrac{z}{2}$ to the unit disc. Define

$$G(w) := \frac{g(2w)}{2}. \tag{3.29}$$

Since g is defined and conformal on $D(0,2)$, then G is defined and conformal on the unit disc and satisfies $G(0) = g(0) = 0$ and $G'(0) = g'(0) = 1$. By the Koebe Distortion Theorem, we have

$$\frac{|w|}{(1+|w|)^2} \leq |G(w)| \leq \frac{|w|}{(1-|w|)^2}. \tag{3.30}$$

It follows by (3.29) and (3.30) that

$$\frac{|z|/2}{(1+|z|/2)^2} \leq \frac{|g(z)|}{2} \leq \frac{|z|/2}{(1-|z|/2)^2}.$$

Hence

$$\frac{4|z|}{(2+|z|)^2} \leq |g(z)| \leq \frac{4|z|}{(2-|z|)^2}.$$

as required. □

In the following Lemma we will give some estimations of the Fatou coordinate on repelling petals with certain properties. We are going to use these estimations in the proof of the expansion property of the euclidean derivative of an entire function near a parabolic point, see Proposition 3.20.

Lemma 3.19. *Let f be as in (3.1) and let κ and F_{R} be defined as in (3.6) and (3.23), respectively. Then for each repelling vector at zero there exists a well-behaved repelling petal $\mathcal{P}_{\mathrm{R}}$ that makes an angle α/p where $\pi/2 < \alpha \leq \pi$, such that if g is the local inverse of f for which $\mathcal{P}_{\mathrm{R}}$ is an attracting petal then*

$$|F_{\mathrm{R}}(w) - 1 - w| < \frac{1}{4}, \qquad w \in \kappa(g(\mathcal{P}_{\mathrm{R}})). \tag{3.31}$$

and the Fatou coordinate at infinity satisfies

$$9/80 < |\Phi'(w)| < 169/48, \qquad w \in \kappa(g(\mathcal{P}_{\mathrm{R}})). \tag{3.32}$$

Proof. Let ψ_{R} be defined as in (3.12). By Lemma 3.13, there exists $M > 0$ such that $\psi_{\mathrm{R}}(S_{-M,\pi})$ is a well-behaved repelling petal that makes an angle π/p at zero. Set

$$\widetilde{\mathcal{P}}_{\mathrm{R}} := \psi_{\mathrm{R}}(S_{-M,\pi}).$$

We conjugate f on the petal $\widetilde{\mathcal{P}}_{\mathrm{R}}$ to a map $F_{\mathrm{R}} \colon S_{-M,\pi} \to \mathbb{C}$ which is given by (3.23). By equation (3.24), we can choose $R \geq M$ such that

$$|F_{\mathrm{R}}(w) - 1 - w| < \frac{1}{4}, \quad \text{for} \quad |w| > R.$$

Since $S_{-R,\pi} \subset \{w \in \mathbb{C} \colon |w| > R\}$ by definition then we have

$$|F_{\mathrm{R}}(w) - 1 - w| < \frac{1}{4}, \quad \text{for} \;\; w \in S_{-R,\pi}. \tag{3.33}$$

Let κ be defined as in (3.6) and let Φ be the Fatou coordinate at infinity defined on $\kappa(\widetilde{\mathcal{P}}_{\mathrm{R}})$. Suppose that $w \in S_{-(R+2),\alpha}$ with $\pi/2 < \alpha \leq \pi$, and define the map $\Psi : D(0,2) \to \mathbb{C}$ as follows

$$\Psi(a) := \frac{\Phi(w+a) - \Phi(w)}{\Phi'(w)}. \tag{3.34}$$

Note that $D(w,2) \subset S_{-R,\pi}$ as $w \in S_{-(R+2),\alpha}$ with $\pi/2 < \alpha \leq \pi$. So, if $a \in D(0,2)$ then

$$|w + a - w| = |a| < 2,$$

and hence $w + a \in S_{-R,\pi}$. Since Φ is conformal on $S_{-M,\pi} = \kappa(\widetilde{\mathcal{P}}_{\mathrm{R}})$ then Ψ is conformal on $D(0,2)$. Moreover, Ψ satisfies that $\Psi(0) = 0$ and $\Psi'(0) = 1$. It follows from Lemma 3.18 that

$$\frac{4|a|}{(2+|a|)^2} \leq |\Psi(a)| \leq \frac{4|a|}{(2-|a|)^2}. \tag{3.35}$$

for all $a \in D(0,2)$.

Since $w \in S_{-(R+2),\alpha} \subset S_{-R,\pi}$ for $\pi/2 < \alpha \leq \pi$ then it follows from (3.33) that

$$3/4 < |F_{\mathrm{R}}(w) - w| < 5/4.$$

Hence, we can put $a = F_{\mathrm{R}}(w) - w$ in (3.34) and (3.35), and we obtain

$$48/169 < \frac{|\Phi(F_{\mathrm{R}}(w)) - \Phi(w)|}{|\Phi'(w)|} < 80/9, \tag{3.36}$$

for all $w \in S_{-(R+2),\alpha}$ and $\pi/2 < \alpha \leq \pi$.

By Lemma 3.13 we can choose the well-behaved repelling petal $\mathcal{P}_{\mathrm{R}}$ to be $\psi_{\mathrm{R}}(S_{-(R+2),\alpha})$ for $\pi/2 < \alpha \leq \pi$. Let g be the local inverse of f for which $\mathcal{P}_{\mathrm{R}}$ is an attracting petal. Then $g(\mathcal{P}_{\mathrm{R}}) \subset \mathcal{P}_{\mathrm{R}}$ by definition. Since $\kappa(g(\mathcal{P}_{\mathrm{R}})) \subset \kappa(\mathcal{P}_{\mathrm{R}}) = S_{-(R+2),\alpha}$ then it follows from equations (3.28) and (3.36) that

$$9/80 < |\Phi'(w)| < 169/48,$$

for all $w \in \kappa(g(\mathcal{P}_{\mathrm{R}}))$.

Note that inequality (3.33) holds for all $w \in \kappa(g(\mathcal{P}_{\mathrm{R}}))$, because $\kappa(g(\mathcal{P}_{\mathrm{R}})) \subset S_{-(R+2),\alpha} \subset S_{-R,\pi}$ for $\pi/2 < \alpha \leq \pi$. $\square$

In the next result we are going to prove that the Euclidean derivative of an entire function near a parabolic point is expanding which is very important for the proof of Proposition 4.11, and hence for the continuity of the semiconjugacy that we are going to construct in Section 5.2. We will consider an entire function f with a parabolic fixed point $\zeta \in \mathbb{C}$ and p repelling(attracting) vectors. For $n \in \mathbb{N}$, we need to find a lower bound of the Euclidean derivative of f^n at a point z which has the property that the point itself and all its preimages under one inverse branch of f lie in a well-behaved repelling petal $\mathcal{P}$ at ζ. We should note that f is univalent on $\mathcal{P}_R(\zeta)$ by definition. First, we will find this lower bound for $\zeta = 0$ by using *Fatou coordinates at infin-*

ity. Then we will find a global lower bound for any parabolic fixed point $\zeta \in \mathbb{C}$.

Proposition 3.20. *Let f be an entire function that has a parabolic fixed point at $\zeta \in \mathbb{C}$ with multiplier one and multiplicity $p+1$. Then there exists $K_\zeta > 0$ with the following property. For each repelling vector at ζ there exists a well-behaved repelling petal $\mathcal{P}$ that makes an angle $\pi/(2p) < \alpha/p \leq \pi/p$ at ζ such that if $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $z_j := f^j(z) \in \mathcal{P}$ for $0 \leq j \leq n$ then the Euclidean derivative of f^n at z satisfies*

$$|(f^n)'(z)| > K_\zeta \; |z_n - \zeta|^{1+p} \; n^{(1+p)/p}. \tag{3.37}$$

Proof. We will first prove the claim for $\zeta = 0$. Let κ, ψ_{R} and F_{R} be defined as in (3.6), (3.12) and (3.23), respectively. Let us pick a well-behaved repelling petal $\mathcal{P}$ at zero with the following properties.

(a) If $\Phi\colon \kappa(\mathcal{P}) \to \mathbb{C}$ is the Fatou coordinate at infinity, then the functions F_{R} and Φ satisfy inequalities (3.31) and (3.32).

(b) If g is the local inverse of f in a neighbourhood of zero for which $\mathcal{P}$ is an attracting petal, then $z_n \in g(\mathcal{P})$.

Observe that such a petal exists for each repelling vector at ζ, and it makes an angle α/p where $\pi/2 < \alpha \leq \pi$ at ζ by Lemma 3.19.

Suppose that $z_j \in \mathcal{P}$ for $0 \leq j \leq n$ and $w = \kappa(z)$ then by (3.23), we have

$$F_{\mathrm{R}}^j(w) = \kappa \circ f^j \circ \psi_{\mathrm{R}}(w), \quad 0 \leq j \leq n. \tag{3.38}$$

Since g is a local inverse of f then $z_j \in g(\mathcal{P})$ for $0 \leq j \leq n-1$. This together with the property (b) of the petal $\mathcal{P}$ implies that

$$\kappa(z_j) = \kappa(f^j(z)) = F_{\mathrm{R}}^j(w) \in \kappa(g(\mathcal{P})), \quad 0 \leq j \leq n. \tag{3.39}$$

By equation (3.28), we have

$$\Phi\left(F_{\mathrm{R}}\left(\xi\right)\right)=\Phi(\xi)+1,$$

for all $\xi\in\kappa(g(\mathcal{P}))$.

So it follows from (3.39) that

$$\Phi\left(F_{\mathrm{R}}(F_{\mathrm{R}}^{j}(w))\right)=\Phi\left(F_{\mathrm{R}}^{j}(w)\right)+1,\quad 0\leq j\leq n-1.$$

Hence, we have

$$\Phi\left(F_{\mathrm{R}}^{n}(w)\right)=\Phi(w)+n.$$

Then the derivative of Φ satisfies

$$\Phi'\left(F_{\mathrm{R}}^{n}(w)\right)\cdot(F_{\mathrm{R}}^{n})'(w)=\Phi'(w)$$

and hence

$$|(F_{\mathrm{R}}^{n})'(w)|=\frac{|\Phi'(w)|}{|\Phi'\left(F_{\mathrm{R}}^{n}(w)\right)|}.$$

By property (a) of the repelling petal $\mathcal{P}$, and since $w,F_{\mathrm{R}}^{n}(w)\in\kappa(g(\mathcal{P}))$ then it follows from (3.32) that

$$|(F_{\mathrm{R}}^{n})'(w)|=\frac{|\Phi'(w)|}{|\Phi'\left(F_{\mathrm{R}}^{n}(w)\right)|}>\frac{9}{80}\cdot\frac{48}{169}=\frac{27}{845}. \tag{3.40}$$

By equations (3.6), (3.12) and (3.23) we have

$$F_{\mathrm{R}}^{n}(w)=\frac{-1}{pa}\left[f^{n}\left(\left(\frac{-1}{paw}\right)^{1/p}\right)\right]^{-p}.$$

Thus,

$$(F_{\mathrm{R}}^n)'(w) = \frac{1}{p^2a^2w^2}\left(\frac{-1}{paw}\right)^{-1+1/p}(f^n)'\left(\left(\frac{-1}{paw}\right)^{1/p}\right)\left(f^n\left(\left(\frac{-1}{paw}\right)^{1/p}\right)\right)^{-(1+p)}. \tag{3.41}$$

It then follows from (3.40) and (3.41) that

$$\left|(f^n)'\left(\left(\frac{-1}{paw}\right)^{1/p}\right)\right| > \frac{27}{845}\ p^2|aw|^2\left|\frac{1}{paw}\right|^{1-1/p}\left|f^n\left(\left(\frac{-1}{paw}\right)^{1/p}\right)\right|^{1+p}.$$

Again by equations (3.6) and (3.12), we obtain

$$|(f^n)'(z)| > \frac{27}{845}\frac{1}{|z^p|^2}|z^p|^{1-1/p}|f^n(z)|^{1+p}.$$

By setting $A := 27/845$, we have

$$|(f^n)'(z)| > A|z|^{-(1+p)}|f^n(z)|^{1+p} = A|z|^{-(1+p)}|z_n|^{1+p}. \tag{3.42}$$

Recall that $\{\kappa(z_j)\colon 0 \le j \le n\} \subset \kappa(g(\mathcal{P}))$. So it follows from (3.31) that

$$|F_{\mathrm{R}}(\kappa(z_j)) - 1 - \kappa(z_j)| < 1/4,$$

for $0 \le j \le n$. Thus, by equation (3.39) and simple calculations we obtain

$$|F_{\mathrm{R}}^n(\kappa(z)) - n - \kappa(z)| < n/4,$$

and hence

$$-\mathrm{Re}(F_{\mathrm{R}}^n(\kappa(z))) + \mathrm{Re}(\kappa(z)) + n < n/4.$$

So it follows that

$$-\mathrm{Re}(\kappa(z_n)) + \mathrm{Re}(\kappa(z)) + n < n/4.$$

Recall that the well-behaved repelling petal $\mathcal{P}$ makes an angle $\pi/2p < \alpha/p \leq \pi/p$ at zero. Thus, by Lemma 3.13 there exists $M > 0$ such that $\mathcal{P} = \psi_{\mathrm{R}}(S_{-M,\alpha})$ where $S_{-M,\alpha}$ is given by (3.5) and $\pi/2 < \alpha \leq \pi$. So the set $\kappa(g(\mathcal{P}))$ is contained in the left half plane $\mathbb{H}_l$ because $\kappa(g(\mathcal{P})) \subset \kappa(\mathcal{P}) = S_{-M,\alpha}$. Since $\kappa(z_n) \in \kappa(g(\mathcal{P}))$ by the property (b) of the petal $\mathcal{P}$, then $\mathrm{Re}(\kappa(z_n)) < 0$. Hence, we have

$$\mathrm{Re}(\kappa(z)) < -3n/4,$$

which implies that

$$|\kappa(z)| > 3n/4.$$

By definition of κ given in (3.6)

$$|z|^{-1} > \left(\frac{3p|a|}{4}\right)^{1/p} n^{1/p}.$$

Set $K := A\left(\frac{3p|a|}{4}\right)^{(1+p)/p}$, then by (3.42) we finally obtain

$$|(f^n)'(z)| > K\ |z_n|^{1+p}\ n^{(1+p)/p}.$$

This proves the claim for $\zeta = 0$.

Now, suppose that $\zeta \neq 0$ is a fixed parabolic point of f with multiplier one and multiplicity $p+1$. We conjugate f to a map $g(w) := f(w+\zeta) - \zeta$. Then g has a parabolic fixed point at zero with multiplier one and multiplicity $p+1$. Then there exists a well-behaved repelling petal $\widetilde{\mathcal{P}}$ that makes an angle $\pi/(2p) < \widetilde{\theta} \leq \pi/p$ for each repelling vector at zero such that, if $w \in \mathbb{C}$ and $w_j := g^j(w) \in \widetilde{\mathcal{P}}$ for $0 \leq j \leq n$ then the Euclidean derivative of g^n at w satisfies

$$|(g^n)'(w)| > K_\zeta\ |w_n|^{1+p}\ n^{(1+p)/p}.$$

Hence, by definition there exists a well-behaved repelling petal $\mathcal{P}$ that makes an angle $\pi/(2p) < \theta \leq \pi/p$ for each repelling vector at ζ such that, if $z \in \mathbb{C}$ and $z_j \in \mathcal{P}$ for $0 \leq j \leq n$ then the Euclidean derivative of f^n at z satisfies

$$|(f^n)'(z)| > K_\zeta \; |z_n - \zeta|^{1+p} \; n^{(1+p)/p},$$

as claimed. $\square$

Chapter 4

Parabolic transcendental functions

In this chapter, we will present the class of our interest, which we call *the class of parabolic transcendental functions*. We will give the definition of a *parabolic* transcendental function f and some of its dynamical properties. Then we are going to construct an open connected neighbourhood W of the Julia set $\mathcal{J}(f)$. We then define a metric σ on W that depends on some constants ϵ and M. We show that there exist ϵ and M for which the function f is expanding with respect to the metric σ. The expanding property of σ is very crucial for the proof of the existence of the semiconjugacy in the next chapter.

4.1 Definition and dynamical properties

Definition 4.1. *A transcendental entire map $f \in \mathcal{B}$ is called* parabolic *if the following hold:*

(a) $P_{\mathcal{J}} := P(f) \cap \mathcal{J}(f)$ *is finite and nonempty.*

(b) *Every point in the set $P_{\mathcal{J}}$ is a parabolic periodic point of f.*

(c) $S(f) \subset \mathcal{F}(f)$.

We will show that the Fatou set of such a function consists only of attracting or parabolic basins. It is never empty because it contains at least one parabolic basin by definition.

Remark. If $P_{\mathcal{J}} = \emptyset$ and (c) holds, then f is hyperbolic. Proofs of all our results work for hyperbolic functions, but they were already established in [Rem09].

Proposition 4.2. *Let $f \in \mathcal{B}$ be parabolic. Then the Fatou set of f consists of finitely many parabolic and attracting basins, where the number of parabolic basins is not zero. Furthermore, every periodic cycle in $\mathcal{J}(f)$ is repelling or parabolic.*

Proof. Let U be a component of the Fatou set. Then U is either wandering, periodic or preperiodic.

First, we are going to show that U cannot be a wandering domain. Suppose that U is a wandering domain. Recall from Section 2.4 that all limit functions of all orbits of points in U are constant and in the set $\mathcal{J}(f) \cap (P(f) \cup \{\infty\})$. By the definition of the Fatou set, if $z \in U$ then the sequence $(f^n(z))_{n\geq 0}$ has a subsequence $(f^{n_j}(z))$ that converges locally uniformly to a point $\zeta \in P_{\mathcal{J}}$ or to ∞.

Claim. If $z \in U$, $\zeta \in P_{\mathcal{J}}$ and $f^{n_j}(z) \to \zeta$ then $f^n(z)$ tends to the periodic cycle of the point ζ.

Proof of claim. Suppose that ζ is a parabolic periodic point of period k and Ξ is its orbit. Let us take a small compact neighbourhood $\mathcal{N}$ of the set Ξ such that $f(\mathcal{N}) \cap P_{\mathcal{J}} \backslash \Xi = \emptyset$. This means that it is impossible to have a limit point of $(f^n(z))_{n\geq 0}$ in the set $f(\mathcal{N}) \backslash \Xi \supseteq f(\mathcal{N}) \backslash \mathrm{int}(\mathcal{N})$.

We will assume, for the sake of contradiction, that for every $n \geq 0$ there exists $m \geq n$ such that $f^m(z) \notin \mathrm{int}(\mathcal{N})$. Since $f^{n_j}(z) \to \zeta$ there exists $j_0 \in \mathbb{N}$ such that $f^{n_j}(z) \in \mathrm{int}(\mathcal{N})$ for all $j \geq j_0$. Then, there exists $m_j \geq n_j$ such that $f^{m_j}(z) \notin \mathrm{int}(\mathcal{N})$. Let m_j be minimal with this property. Then we have that $f^{m_j - 1}(z) \in \mathcal{N}$ and hence $f^{m_j}(z) \in f(\mathcal{N}) \backslash \mathrm{int}(\mathcal{N})$. It follows that the set

$f(\mathcal{N})\backslash\mathrm{int}(\mathcal{N})$ contains a limit point of $(f^n(z))_{n\geq 0}$, which is a contradiction. This proves that the orbit of $z \in U$ converges to the periodic cycle of $\zeta \in P_{\mathcal{J}}$, as claimed. $\triangle$

By the claim above if $f^{n_j}(z) \to \zeta$ then z is in the parabolic basin of ζ and hence U is preperiodic, which is not the case as U is wandering.

Now suppose that $z \in U$ and $f^{n_j}(z) \to \infty$. Since f is in class $\mathcal{B}$ then we cannot have $f^n(z) \to \infty$ by Theorem 2.8. Thus, there must be another subsequence $(f^{m_j}(z))$ of $(f^n(z))_{n\geq 0}$ and a point $\omega \in P_{\mathcal{J}}$ such that $f^{m_j}(z) \to \omega$. It follows again by the claim above that z is in the parabolic basin of ω, which is again impossible as U is a wandering domain.

By Theorem 2.10, if U is periodic then it can be an immediate basin of an attracting or parabolic periodic point, a Siegel disc, or a Baker domain. If U is a Siegel disc then $\partial U \subset P(f)$ by Theorem 2.11. But the boundary of a Siegel disc is contained in the Julia set, which means that $\partial U \subset P_{\mathcal{J}}$. This is impossible because $P_{\mathcal{J}}$ is finite, so f has no Siegel disc. As mentioned above, we cannot have $f^n|_U \to \infty$ and so U cannot be a Baker domain. Hence the Fatou set is the union of all attracting and parabolic basins.

If U is preperiodic, then it will be eventually mapped to a periodic Fatou component. By the discussion above, U is contained in either an attracting basin or in a parabolic basin.

Since f is in class $\mathcal{B}$ and $S(f) \subset \mathcal{F}(f)$ by definition then $S(f) \cap \mathcal{F}(f) = S(f)$ is compact. The union of attracting and parabolic components forms an open cover of $S(f)$. Hence, there are finitely many such components. Every attracting or parabolic cycle must contain at least one point of $S(f)$ by Theorem 2.11. Hence there are finitely many such basins which together make up the Fatou set. Again by definition, $P_{\mathcal{J}}$ is not empty and consists only of parabolic points. So there is at least one parabolic component in the Fatou set.

If $z \in U$ is an irrationally indifferent periodic point (Cremer point), then it follows from [Mil06, Lemma 11.1] and Theorem 2.7 that z belongs to the set $P_{\mathcal{J}}(f)$. However, the set $P_{\mathcal{J}}(f)$ contains only parabolic periodic points of f. Hence there are no Cremer points. $\square$

By definition the set of postsingular values $P(f)$ is closed. In the following Lemma we will show that for a parabolic transcendental function this set is bounded and hence compact.

Lemma 4.3. *Let $f \in \mathcal{B}$ be parabolic, then the postsingular set $P(f)$ is compact.*

Proof. Note that $S(f) \subset \mathcal{F}(f)$ by Definition 4.1, and recall that $\mathcal{F}(f)$ is a finite union of attracting and parabolic basins. Hence the orbit of each point $s \in S(f)$ converges to an attracting or parabolic cycle. Suppose that U is a union of open neighbourhoods of every attracting point of f, and $\mathcal{P}_A(\zeta)$ is a union of simply connected bounded attracting petals (Theorem 3.8) for every attracting vector at every point $\zeta \in \operatorname{Par}(f)$. Set $\mathcal{P}_A := \bigcup_{\zeta \in \operatorname{Par}(f)} \mathcal{P}_A(\zeta)$. For each $s \in S(f)$ we will choose a small disc D_s centered at s such that $D_s \subset \mathcal{F}(f)$. It follows from the definition of the Fatou set that there exists $N_s \geq 0$ such that $\overline{f^n(D_s)} \subset U \cup \mathcal{P}_A$ for all $n > N_s$ and $s \in S(f)$. Observe that the union of these discs forms an open cover of the set $S(f)$. Since $S(f)$ is compact then there exists a finite subcover of $S(f)$. This means that there are some points $s_1, \ldots, s_n \in S(f)$ and discs $D_1, \cdots, D_n$ such that D_k is centered at s_k and that $S(f) \subset \bigcup_{i=1}^n D_i$. For each $k = 1, \ldots, n$ we define $G_k := D_k \cup f(D_k) \cup \ldots \cup f^{N_s}(D_k)$ which is bounded because it is a finite union of bounded sets. This implies that the set $G := (\bigcup_{k=1}^n \overline{G}_k) \cup \overline{U} \cup \overline{\mathcal{P}}$ is also bounded and satisfies $P(f) \subset G$. Thus the set $P(f)$ is bounded, and hence compact as it is closed by definition. $\square$

Recall that A_{Att} and A_{Par} denote the sets of all points whose orbits converge to an attracting or parabolic orbit of f, respectively, where the parabolic points themselves do not belong to A_{Par}. Let $D_\epsilon(A)$ denote the Euclidean neighbourhood of a set $A \subset \mathbb{C}$.

Note that if C is a compact subset of A_{Att} then $\overline{\bigcup_{k\geq 0} f^k(C)}$ is bounded and hence a compact subset of A_{Att}. Similarly, if $C \subset A_{\mathrm{Par}}$ is compact then $\overline{\bigcup_{k\geq 0} f^k(C)}$ is a compact subset of $A_{\mathrm{Par}} \cup \mathrm{Par}(f)$. The boundness of the set $\overline{\bigcup_{k\geq 0} f^k(C)}$ can be proved by using the same argument we used in the proof of Lemma 4.3 as the sets A_{Att} and A_{Par} are contained in the Fatou set.

Proposition 4.4. [MB12, Proposition 2.6] *Let f be a nonlinear entire function. If $C \subset A_{Att}$ is a compact set then there exist pairwise disjoint Jordan domains $U_1, ..., U_n$ and $\epsilon > 0$, such that if $K := \bigcup_{k\geq 0} f^k(C)$ and $U := \bigcup_{i=1}^n U_i$ then U has the following properties:*

(a) $\overline{U} \subset A_{Att}$.

(b) $D_\epsilon(\overline{K}) \subset U$.

(c) $\overline{f(U)} \subset U$.

The following proposition is similar to [MB09, Proposition 3.2] but contains additional details which are required in our settings. We shall use the following stronger statement.

Proposition 4.5. *Let f be a nonlinear entire function. If $C \subset A_{\mathrm{Par}}$ is a compact set then there exist bounded simply connected pairwise disjoint domains $U_1, ..., U_n$ such that if $K := \bigcup_{k\geq 0} f^k(C)$ and $U := \bigcup_{i=1}^n U_i$ then U has the following properties:*

(a) $\overline{U} \backslash \mathrm{Par}(f) \subset A_{\mathrm{Par}}$.

(b) *If ζ is a parabolic point such that there exists a point $z \in C$ whose orbit converges to the orbit of ζ then $\zeta \in \partial U$.*

(c) *Let* p_ζ *be the number of attracting vectors at a parabolic point* $\zeta \in \partial U$. *Then for every attracting vector at* ζ *there exists a union of well-behaved attracting petals* $\mathcal{F}_{\mathrm{A}}(\zeta)$ *that is contained in* U, *such that each attracting petal* $\mathcal{P}_{\mathrm{A}} \subset \mathcal{F}_{\mathrm{A}}(\zeta)$ *makes an angle* $3\pi/2p_\zeta$ *at* ζ.

(d) *Suppose that* $\mathcal{P}_{\mathrm{R}}$ *is a well-behaved repelling petal that has angle* $\theta > \pi/2p_\zeta$ *at* ζ. *Then* $\mathcal{P}_{\mathrm{R}} \cap U \neq \emptyset$.

(e) $\overline{K} \subset U \cup \mathrm{Par}(f)$.

(f) $f(\overline{U}) \subsetneq \overline{U}$ *and* $f\left(\overline{U} \backslash \mathrm{Par}(f)\right) \subsetneq U$.

Proof. The components of A_{Par} form an open cover of the set C. Since C is compact then it has a finite subcover. Note that each component of A_{Par} is either periodic or preperiodic by definition. Hence there are finitely many components $A_1, \cdots, A_n$ of A_{Par} such that $K \subset \bigcup_{i=1}^{n} A_i$ and $K_i := K \cap A_i \neq \emptyset$ for all $i = 1, \ldots, n$.

If A_i is strictly preperiodic then the set K_i is compact. However, if A_i is periodic and ζ_i is the unique parabolic periodic point on the boundary of A_i then ζ_i belongs to $\overline{K}$ by definition, in particular, $\zeta_i \in \partial\overline{K}$ but $\zeta_i \notin A_i$. Since all the orbits of the points in K_i converge to the orbit of ζ_i then $K_i \cup \{\zeta_i\}$ is compact set.

We can assume, without loss of generality, that there is only one parabolic cycle of a periodic point ζ that intersects $\overline{K}$. Otherwise, we can do the same procedure for each parabolic cycle independently. Let us also assume that the parabolic point ζ is fixed with multiplier one because otherwise we can pass to an iterate of f. Let p_ζ be the number of attracting vectors at ζ. It is required for the union of attracting and repelling petals to cover a neighbourhood of ζ. By Lemma 3.12 and Remark 3.14, we can pick a union of well-behaved attracting petals $\mathcal{F}_{\mathrm{A}}(\zeta) := \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_p$ for every attracting vector at ζ such that each attracting petal $\mathcal{P}_i$ makes an angle $3\pi/2p_\zeta$ at ζ. Additionally, if f has more than one parabolic point, say ζ_1 and ζ_2, then we choose the union of well-behaved attracting petals at ζ_1 and ζ_2 so that their closures are disjoint.

Suppose that $A_1, \ldots, A_p$ are the periodic components of A_{Par} such that $\mathcal{P}_i \subset A_i$ for $i = 1, \ldots, p$. We will first consider the components $A_{p+1}, \cdots, A_n$ of A_{Par} that are not periodic. For simplicity, let us assume that there is only one non-periodic chain of parabolic basins, i.e. $A_n \xrightarrow{f} A_{n-1} \xrightarrow{f} \cdots\cdots \xrightarrow{f} A_{p+1} \xrightarrow{f} A_p$. If this is not the case then we can repeat the same procedure for each such chain. Recall that the intersection K_i for $i = p+1, \ldots, n$ is a compact set. Thus, there is a neighbourhood D_j of K_j such that $\overline{D}_j \subset A_j$ for $j = p+1, \ldots, n$ and that $\overline{f(D_n)} \subset D_{n-1}, \cdots\cdots, \overline{f(D_{p+2})} \subset D_{p+1}$.

Now, we will look at the periodic components $A_1, \cdots, A_p$. Recall that ζ is a fixed point with multiplier one and thus each periodic component A_i is associated to one attracting vector at ζ and satisfies $f(A_i) \subset A_i$ [Mil06, Theorem 16.1]. Since the orbit of each point in K_i converges to the parabolic fixed point ζ, then by the definition of attracting petals there exists $\widetilde{N}_i \geq 0$ such that $f^n(K_i) \subset \mathcal{P}_i$ for $n > \widetilde{N}_i$. The set $K_i \backslash \mathcal{P}_i$ is a compact subset of A_i because K_i is compact and $\mathcal{P}_i$ is open. Hence, there is a neighbourhood D_i of $K_i \backslash \mathcal{P}_i$ such that $\overline{D}_i \subset A_i$. We will assume that $D_i \cap f(D_i) \neq \emptyset$. If $i = p$ we will assume additionally that $\overline{f(D_{p+1})} \subset D_p$ because of our assumption above that $A_{p+1} \xrightarrow{f} A_p$. Since $\overline{D}_i$ is a compact subset of A_i then again by definition, there exists $N_i \geq 0$ such that $\overline{f^n(D_i)} \subset \mathcal{P}_i$ for all $n > N_i$. Let N_i be minimal with this property then define

$$F_i := \left(\bigcup_{j=0}^{N_i} f^j\left(D_i\right) \right) \bigcup \mathcal{P}_i.$$

By construction F_i is a connected set with a unique parabolic point on its boundary. The set F_i has also the properties $K_i \subset F_i$, $\overline{F}_i \backslash \mathrm{Par}(f) \subset A_i$,

$f(\overline{F}_i) \subsetneq \overline{F}_i$ and $f(\overline{F}_i \backslash \mathrm{Par}(f)) \subsetneq F_i$.

Observe that the sets F_i and D_k are connected for $i = 1, \ldots, p$ and $k = p+1, \ldots, n$.

Define

$$F := \left(\bigcup_{i=1}^{p} F_i \right) \cup \left(\bigcup_{k=p+1}^{n} D_k \right),$$

then by construction we have $\overline{K} \subset F \cup \mathrm{Par}(f)$. Note also by construction that F has the properties $f(\overline{F}) \subsetneq \overline{F}$ and $f(\overline{F} \backslash \mathrm{Par}(f)) \subsetneq F$.

By construction again the components of F are not necessarily simply connected. Let V be the union of the bounded components of $\mathbb{C} \backslash F$ then set $U := F \cup \overline{V}$. Note that $\partial V \cap \mathrm{Par}(f) = \emptyset$. Since f is entire and $f(\overline{F} \backslash \mathrm{Par}(f)) \subsetneq F$ then we have

$$\partial f(V) \subset f(\partial V) \subset f(\partial F \backslash \mathrm{Par}(f)) \subsetneq F,$$

and hence $f(\overline{V}) \subsetneq F$.

Now let $U_1, \ldots, U_n$ be the components of U. By construction each of these components is bounded and simply connected. Since $\overline{F}_i \backslash \mathrm{Par}(f) \subset A_i$ and $\overline{D}_k \subset A_k$ for $i = 1, \ldots, p$ and $k = p+1, \ldots, n$, then $\overline{U}_i \backslash \mathrm{Par}(f) \subset A_i$ for $i = 1, \ldots, n$. Hence, we have $\overline{U} \backslash \mathrm{Par}(f) \subset A_{\mathrm{Par}}$. Since $C \subset A_{\mathrm{Par}}$ then the orbit of each point $z \in C$ converges to an orbit of a parabolic point. By construction, every parabolic point with such property is on the boundary of U.

Once again by construction, there is a union of well-behaved attracting petals $\mathcal{F}_{\mathrm{A}}(\zeta)$ for every attracting vector at ζ, that is contained in U. Each attracting petal $\mathcal{P}_i \subset U$ makes an angle $3\pi/2p_\zeta$ at ζ. Recall that the attracting and repelling petals are arranged symmetrically around any parabolic point. Then every well-behaved repelling petal with angle $\theta > \pi/2p_\zeta$ at ζ must intersect with two well-behaved attracting petals in $\mathcal{F}_{\mathrm{A}}(\zeta)$, and hence intersect with U.

Since $U = F \cup \overline{V}$ then it follows from the above properties of F that U has the properties $\overline{K} \subset U \cup \mathrm{Par}(f)$, $f(\overline{U}) \subsetneq \overline{U}$ and $f\left(\overline{U} \backslash \mathrm{Par}(f)\right) \subsetneq U$, as required. $\square$

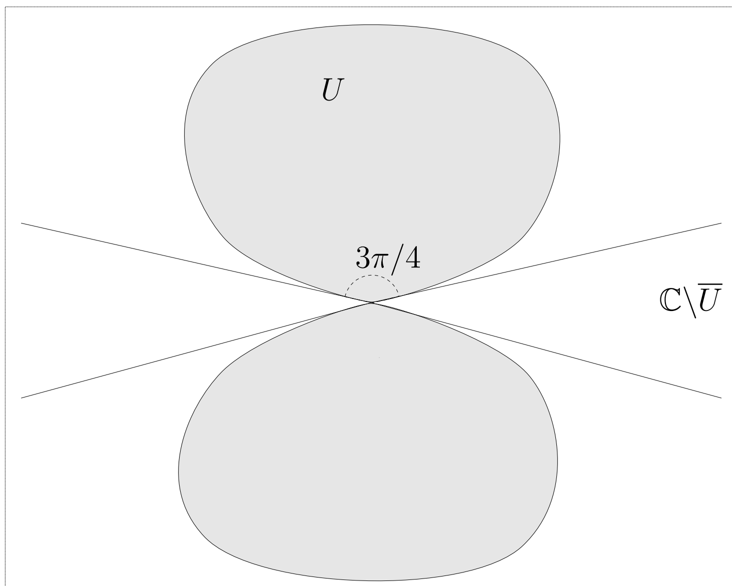

Figure 4.1: Illustration of the choice of two well-behaved attracting petals as in the proof of Proposition 4.6

Remark. In the proof of Proposition 4.5 we picked a union of well-behaved attracting petals with certain size and this union is contained in U. This means that there is a "thin" component centered in each repelling vector that is contained in $\mathbb{C}\backslash\overline{U}$, see Figure 4.1. This will allow us to apply Lemma 3.6 in the proof of Lemma 4.7 which is very useful tool in the proof of the expanding property in Theorem 4.10.

4.2 Constructing a neighbourhood W of the Julia set

We will construct an open neighbourhood of the Julia set of a parabolic transcendental entire function f. Note that by Proposition 4.2, the Fatou set of such a function consists only of attracting and parabolic basins.

Proposition 4.6. *Let $f \in \mathcal{B}$ be parabolic. Then there exists an open connected set W that has the following properties:*

(a) $U := \mathbb{C}\backslash\overline{W}$ *is a finite union of bounded simply connected domains with the properties* $f(\overline{U}\backslash\mathrm{Par}(f)) \subsetneq U$ *and* $S(f) \subset U$.

(b) *Let p_ζ be the number of attracting vectors at any parabolic point* $\zeta \in \mathrm{Par}(f)$. *If $\mathcal{F}_{\mathrm{R}}(\zeta)$ is a union of well-behaved repelling petals each of which makes an angle $\theta > \pi/2p_\zeta$ at ζ, then there exists $r_\zeta > 0$ such that each component of $D(\zeta, r_\zeta) \cap W$ is contained in one repelling petal in $\mathcal{F}_{\mathrm{R}}(\zeta)$. Moreover, each of these components is contained in a sector of angle $\pi/2p_\zeta$.*

(c) $\mathcal{J}(f)\backslash\mathrm{Par}(f) \subset W$.

(d) $V := f^{-1}(W) \subsetneq W$.

(e) $\mathrm{Par}(f) \subset \partial U = \partial W$ *and* $\partial V \cap \partial W = \mathrm{Par}(f)$.

Proof. First we are going to construct the set U, then we will define W to be the complement of U in $\mathbb{C}$. The set $S(f)$ is closed by definition and bounded since $f \in \mathcal{B}$, and hence compact. Since f is parabolic then $S(f) \subset \mathcal{F}(f)$. Recall from Proposition 4.2 that $\mathcal{F}(f)$ is a finite union of attracting and parabolic basins. Let U_i be as in the Propositions 4.4 and 4.5 relative to the set $S(f) \cap (A_{\mathrm{Att}} \cup A_{\mathrm{Par}})$. So the sets U_i are simply connected bounded pairwise disjoint and $\overline{U}_i \subset A_{\mathrm{Att}} \cup A_{\mathrm{Par}} \cup \mathrm{Par}(f)$ for all i. Replacing, U_i by $f(U_i)$

if necessary, we can assume that $\overline{U}_i$ and $\overline{U}_j$ intersect only if $U_i, U_j \subset A_{\text{Par}}$ and then the intersection is a single point in $\text{Par}(f)$. Set $U := \bigcup_{i=1}^{n} U_i$ and $W := \mathbb{C}\backslash\overline{U}$. Then W is open and connected.

It also follows from Proposition 4.4 and Proposition 4.5 that

$$\mathcal{P}(f) \subset U \cup \text{Par}(f), \tag{4.1}$$

where $\overline{U}\backslash\text{Par}(f) \subset A_{\text{Att}} \cup A_{\text{Par}}$, and U has the properties $f(\overline{U}) \subsetneq \overline{U}$ and $f(\overline{U}\backslash\text{Par}(f)) \subsetneq U$. Since $S(f) \subset \mathcal{P}(f)$ and $S(f) \cap \text{Par}(f) = \emptyset$ by definition then it follows from (4.1) that $S(f) \subset U$.

Let us prove now (b). For every parabolic point $\zeta \in \mathbb{C}$, let p_ζ be the number of attracting petals at ζ. Let $\mathcal{F}_{\text{A}}(\zeta)$ be the union of well-behaved attracting petals given in Proposition 4.5(c). Recall that repelling petals are arranged symmetrically around ζ. Note that $\mathcal{F}_{\text{A}}(\zeta)$ consists of p_ζ well-behaved attracting petals each of which makes an angle $3\pi/2p_\zeta$ at ζ. So, if $\mathcal{F}_{\text{R}}(\zeta)$ is a union of well-behaved repelling petals each of which makes an angle $\theta > \pi/2p_\zeta$ at ζ, then $\mathcal{F}_{\text{R}}(\zeta) \cap \mathcal{F}_{\text{A}}(\zeta) \neq \emptyset$. Furthermore, there exists $r_\zeta > 0$ such that $D(\zeta, r_\zeta) \cap W$ is disconnected set with p_ζ components. This proves the properties (b) of the set W.

Recall that $\mathcal{F}(f)$ consists only of attracting and parabolic basins. Since $\overline{U}\backslash\text{Par}(f) \subset A_{\text{Att}} \cup A_{Par}$ then $\overline{U} \subset \mathcal{F}(f) \cup \text{Par}(f)$. Hence, $\mathcal{J}(f)\backslash\text{Par}(f) \subset W$. Since $f(\overline{U}) \subsetneq \overline{U}$ then

$$V = f^{-1}(W) = f^{-1}(\mathbb{C}\backslash\overline{U}) \subsetneq \mathbb{C}\backslash\overline{U} = W.$$

We will now prove that $(\partial W \cap \partial V) = \text{Par}(f)$. First, we are going to prove that $(\partial W \cap \partial V) \subset \text{Par}(f)$. Since $f(\overline{U}\backslash\text{Par}(f)) \subset U$ then $\overline{U}\backslash\text{Par}(f) \subset f^{-1}(U)$.

Thus, we have

$$W \cup \mathrm{Par}(f) \supset \mathbb{C} \backslash f^{-1}(U) = f^{-1}(\mathbb{C} \backslash U) = f^{-1}(\overline{W}) = \overline{V}.$$

Hence $(\partial W \cap \partial V) \subset \mathrm{Par}(f)$. Since we set $C = S(f) \cap A_{\mathrm{Par}}$ in Proposition 4.5 and because each parabolic basin contains a singular value of f, then it follows from Proposition 4.5(b) that $\mathrm{Par}(f) \subset \partial U = \partial W$. Thus, by the invariance of the set of parabolic points we have $\mathrm{Par}(f) \subset \partial V$, and hence $\mathrm{Par}(f) \subset \partial W \cap \partial V$. This proves that W satisfies the properties in the statement. $\square$

In the following Lemma we will show more properties of the neighbourhood W of the Julia set we constructed in Proposition 4.6. These properties are useful in the study of the expanding property of f with respect to a new metric which we are going to define in the next section.

Lemma 4.7. *Let $f\colon V \to W$ be as in Proposition 4.6. Suppose that $\zeta \in \mathbb{C}$ is a parabolic fixed point of f with multiplier one then, there exists $r_\zeta > 0$ such that the following conditions are satisfied for all $r \leq r_\zeta$.*

1. *The disc $D(\zeta, r_\zeta)$ does not contain any singular value of f.*

2. *Let V_{ζ, r_ζ} denote the connected component of $f^{-1}(D(\zeta, r_\zeta))$ that contains ζ. Then $|f(z) - \zeta| > |z - \zeta|$, and $|f'(z)| > 1$ for all $z \in \big(V_{\zeta,r_\zeta} \cup D(\zeta, r_\zeta)\big) \cap W$.*

3. *If $z \in W$ and $|f(z) - \zeta| \leq |z - \zeta|$ then $z \notin V_{\zeta, r_\zeta}$.*

4. *The set $f^{-1}(D(\zeta, r_\zeta)) \backslash V_{\zeta, r_\zeta}$ is contained in W. In particular, $f^{-1}(D(\zeta, r)) \backslash V_{\zeta, r} \subseteq f^{-1}(D(\zeta, r_\zeta)) \backslash V_{\zeta, r_\zeta} \subset W$.*

Proof. Note that $\mathrm{dist}(\zeta, S(f)) > 0$ because $\zeta \in \mathcal{J}(f)$ by Lemma 3.16, and $S(f) \subset F(f)$ by definition. So it is clear that condition (1) holds whenever

$r_\zeta < \operatorname{dist}(\zeta, S(f))$.

Let p_ζ be the number of repelling vectors at ζ. Let $F_{\mathrm{R}}(\zeta)$ be a union of well-behaved repelling petals at ζ, such that each petal in $F_{\mathrm{R}}(\zeta)$ makes an angle $\alpha > \pi/(2p_\zeta)$ at ζ. Then it follows from Proposition 4.6(b) that there exists $\widetilde{r} > 0$ such that each component of $D(\zeta, \widetilde{r}) \cap W$ is contained in one repelling petal in $F_{\mathrm{R}}(\zeta)$. It also follows from Proposition 4.6(b) that each component of the set $D(\zeta, \widetilde{r}) \cap W$ is contained in a sector of angle $\pi/2p_\zeta$. Thus, by Lemma 3.6 we can choose $\widetilde{r}$ sufficiently small such that

$$|f(z) - \zeta| > |z - \zeta| \quad \text{and} \quad |f'(z)| > 1, \quad z \in D(\zeta, \widetilde{r}) \cap W.$$

Let us choose $0 < r_\zeta < \widetilde{r}$ such that $V_{\zeta, r_\zeta} \subset D(\zeta, \tilde{r})$. Then, we have

$$\left(V_{\zeta, r_\zeta} \cup D(\zeta, r_\zeta)\right) \cap W \subset D(\zeta, \tilde{r}) \cap W.$$

Hence, $|f(z) - \zeta| > |z - \zeta|$ and $|f'(z)| > 1$ for all $z \in \left(V_{\zeta, r_\zeta} \cup D(\zeta, r_\zeta)\right) \cap W$.

We will show now that condition (3) can be achieved. Suppose that $z \in W$ and $|f(z) - \zeta| \leq |z - \zeta|$ then by (2), $z \notin \left(V_{\zeta, r_\zeta} \cup D(\zeta, r_\zeta)\right)$ and hence $z \notin V_{\zeta, r_\zeta}$.

To see that condition (4) holds for some $r_\zeta > 0$, note by Proposition 4.6(a) that $U = \mathbb{C} \backslash \overline{W}$ is a finite union of bounded domains. Thus, $\overline{U}$ is a compact set. Let $0 < R < \operatorname{dist}(\zeta, S(f))$. Since the boundary of $D(\zeta, R)$ is locally connected then by [BRG18, Lemma 2.1], there are at most finitely many components $V_0, V_1, \cdots, V_k$ of $f^{-1}(D(\zeta, R))$ intersecting $\overline{U}$. Since each V_j is mapped conformally to $D(\zeta, R)$ then there is a unique preimage of ζ in each component V_j. Let $w_j \in V_j$ where $f(\zeta) = w_j$ for $j = 0, \ldots, k$. Set $w_0 = \zeta \in V_0$. Since $\zeta \in \partial W = \partial U$ and $f(\overline{U} \backslash \operatorname{Par}(f)) \subsetneq U$ by Proposition 4.6(a),(e), then w_j belongs to W for $j > 0$. Recall that the sets $V_0, V_1, \cdots, V_k$ intersect $\overline{U}$, and U is a finite union of bounded simply connected domains.

Thus, we can pick $0 < r_\zeta < R$ sufficiently small such that each component $f^{-1}(D(\zeta, r_\zeta))$ that contains w_j for $1 \leq j \leq k$ does not intersect $\overline{U}$. Since V_{ζ,r_ζ} is the component of $f^{-1}(D(\zeta, r_\zeta))$ containing ζ, we have

$$f^{-1}(D(\zeta, r_\zeta))\backslash V_{\zeta,r_\zeta} \subset \cup_{j=1}^{k} \left(V_j \cap f^{-1}(D(\zeta, r_\zeta))\right) \subset W,$$

as required. □

4.3 Expanding metric on W

In this section we will study the expansion property of a parabolic transcendental map f on W with respect to a metric σ which we are going to define later in this section. This property is significantly important for the proof of Theorem 5.7.

We will first give a result by Rempe in [Rem09], which we are going to use in the proof of the first Lemma in this section.

Lemma 4.8. [Rem09, Lemma 2.1] *Let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence of points in $\mathbb{C}$, with $w_j \to \infty$, satisfying $|w_{j+1}| \leq C|w_j|$ for some constant $C > 1$ and all sufficiently large $j \in \mathbb{N}$. Set $V := \mathbb{C}\backslash\{w_j : j \in \mathbb{N}\}$. Then $1/\rho_V(z) = O(|z|)$ as $z \to \infty$.*

Lemma 4.9. *Let $f: V \to W$ be parabolic and let ρ_W be the hyperbolic metric on W. Suppose that $\zeta \in \operatorname{Par}(f)$ and $\delta > 0$. Then the derivative of f with respect to the hyperbolic metric ρ_W satisfies*

$$\inf_{z\in V, |z-\zeta|>\delta} \|\mathrm{D}f(z)\|_W > 1.$$

Proof. By definition, we have

$$\|\mathrm{D}f(z)\|_V^W = \frac{\rho_W(f(z))}{\rho_V(z)} \cdot |f'(z)|,$$

and

$$\|\mathrm{D}f(z)\|_W = \frac{\rho_W(f(z))}{\rho_W(z)} \cdot |f'(z)|.$$

Hence

$$\|\mathrm{D}f(z)\|_W = \frac{\rho_V(z)}{\rho_W(z)} \cdot \|\mathrm{D}f(z)\|_V^W.$$

Since $f\colon V \to W$ is a covering map by Proposition 4.6(a) then by Pick's theorem

$$\|\mathrm{D}f(z)\|_V^W = 1, \qquad \text{for } z \in V.$$

Additionally, since $V \subset W$ then $\rho_V(z) > \rho_W(z)$ for all $z \in V$. Thus, we have

$$\|\mathrm{D}f(z)\|_W = \frac{\rho_V(z)}{\rho_W(z)} > 1, \qquad \text{for } z \in V. \tag{4.2}$$

Note that $\mathrm{Par}(f) \subset \partial V$ by Proposition 4.6(e). Fix $\delta > 0$ and set

$$\widetilde{V} := \left(\overline{V} \cap \{z \in \mathbb{C}\colon |z - \zeta| \geq \delta, \zeta \in \mathrm{Par}(f)\}\right) \cup \{\infty\}.$$

Define a function $\eta : \widetilde{V} \to (1, \infty]$

$$\eta(z) := \begin{cases} \|\mathrm{D}f(z)\|_W, & \text{if } z \in V, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.3}$$

Note that the sets V and W have common finite boundary points only at the parabolic points of f by Proposition 4.6(e). So since the hyperbolic density of V tends to infinity near ∂V and $V \subset W$, then we have $\frac{\rho_V(z)}{\rho_W(z)} \to \infty$ as $z \to \xi$ where $\xi \in \partial V \backslash \mathrm{Par}(f)$.

We will prove now that $\frac{\rho_V(z)}{\rho_W(z)} \to \infty$ as $z \to \infty$. Let us first estimate the hyperbolic metric on W. By Proposition 4.6(a), the set $\overline{U} = \mathbb{C} \backslash W$ is bounded and hence we can define $R := \max_{z \notin W} |z| + 1$. Set $W' := \mathbb{C} \backslash \overline{D(0,R)}$, then by (2.5) the hyperbolic density on W' is given by

$$\rho_{W'}(z) = \frac{1}{|z| \log(|z|/R)}. \tag{4.4}$$

Since $W' \subset W$ then again by Pick's theorem

$$\rho_W(z) < \rho_{W'}(z). \tag{4.5}$$

Now we estimate the hyperbolic metric on V. Fix some point $w \in \mathbb{C} \backslash W$ such that w belongs to the unbounded component of $\mathbb{C} \backslash S(f)$. Since $w \notin S(f)$ then it cannot be a Picard exceptional point of f. Thus, w has infinitely many preimages under f. Those preimages are contained in $\overline{U} \cup (W \backslash V)$, but since $\overline{U}$ is compact then all but finitely many of them are in $W \backslash V$.

Claim. There exists a sequence $(w_j)_{j \geq 0} \in W \backslash V$ and a constant $K > 1$ such that

$$|w_{j+1}| \leq K|w_j| \quad \text{and} \quad f(w_j) = w, \qquad \text{for all } j \geq 0.$$

Proof of claim. A similar argument was given in [Rem09, proof of Lemma 5.1] and [MB10, proof of Proposition 3.4]. For alternative approach, see [BRG18, proof of Proposition 3.1].

Let γ be a Jordan curve, such that the bounded component of $\mathbb{C} \backslash \gamma$ contains $S(f)$ but not w, and let U' be the unbounded component of $\mathbb{C} \backslash \gamma$. Then $f^{-1}(U')$ is a countable union of simply connected unbounded components (tracts) T_i. Then $f \colon T_i \to U'$ is a universal covering for every $i \geq 0$.

Now, let us pick a tract T_0. Since $U' \cup \{\infty\}$ is a simply connected set in $\widehat{\mathbb{C}}$ then by the Riemann Mapping Theorem, there is a conformal map ϕ which maps U' onto the punctured unit disc $\mathbb{D}^*$. This implies that $\phi \circ f : T_0 \to \mathbb{D}^*$ is a covering map. Thus by a result on covering maps [For91, Theorem 5.10], there is a conformal map ψ which maps T_0 to the left half plane $\mathbb{H}_l$ such that $\phi \circ f = \exp \circ \psi$. Clearly the map $\phi \circ f$ is a covering map of infinite degree.

Since $w \in U' \backslash S(f)$ then w has infinitely many preimages under f in T_0. Suppose that $w_i \in T_0$ is a preimage of w under f. Then $\psi(w_i) \in \mathbb{H}_l$ is a preimage of $\phi(w)$ under the exponential map. Note that the points $\psi(w_i)$ are $2\pi i$ apart. Since the hyperbolic density of the left half plane is given by

$$\rho_{\mathbb{H}_l}(z) = \frac{-1}{\mathrm{Re} z},$$

then the distance $\mathrm{d}_{\mathbb{H}_l}(\psi(w_j), \psi(w_{j+1}))$ for $j \geq 0$ is constant. Since ψ is a conformal map between T_0 and the left half plane $\mathbb{H}_l$ then $\mathrm{d}_{\mathbb{H}_l}(\psi(w_j), \psi(w_{j+1})) = \mathrm{d}_{T_0}(w_j, w_{j+1})$, and hence the distance $\mathrm{d}_{T_0}(w_j, w_{j+1})$ for $j \geq 0$ is also constant.

We can assume without loss of generality that $0 \notin T_0$. Then by the standard estimate on the hyperbolic metric on T_0, we have

$$\rho_{T_0}(z) \geq \frac{1}{2\mathrm{d}_{T_0}(z, \partial T_0)} \geq \frac{1}{2|z|}, \qquad \text{for } z \in T_0$$

Set $\ell := \mathrm{d}_{T_0}(w_j, w_{j+1})$, then by definition we have

$$\ell = \inf_{\widetilde{\gamma}} \int_{a_{\widetilde{\gamma}}}^{b_{\widetilde{\gamma}}} \rho_{T_0}(\gamma(t)) \cdot |\gamma'(t)| \, dt \geq \inf_{\widetilde{\gamma}} \int_{a_{\widetilde{\gamma}}}^{b_{\widetilde{\gamma}}} \frac{|\gamma'(t)|}{2|\gamma(t)|} \, dt \geq \frac{1}{2} \big| \log |w_{j+1}| - \log |w_j| \big|,$$

where $\widetilde{\gamma} : [a_{\widetilde{\gamma}}, b_{\widetilde{\gamma}}] \to T_0$ is any rectifiable curve that connects w_j and w_{j+1}. This means that

$$2\ell \geq \log \frac{|w_{j+1}|}{|w_j|},$$

and hence

$$|w_{j+1}| \leq e^{2\ell} |w_j|.$$

Since $l > 0$ then we can choose $K := e^{2\ell} > 1$, and the claim follows. $\triangle$

So it follows from the claim above and Lemma 4.8, that the hyperbolic density of

$$V' := \mathbb{C}\backslash\{w_j : j \in \mathbb{N}\}$$

satisfies

$$1/\rho_{V'}(z) = O(|z|), \qquad \text{as } z \to \infty.$$

This means that there exist $C > 0$ and $M > 0$ such that

$$1/\rho_{V'}(z) < C|z|, \qquad \text{for } \ |z| > M. \tag{4.6}$$

Since $V \subset V'$ then by Pick's theorem

$$\rho_V(z) > \rho_{V'}(z). \tag{4.7}$$

By equations (4.4), (4.5), (4.6), and (4.7) we obtain

$$\frac{\rho_V(z)}{\rho_W(z)} > \frac{\rho_{V'}(z)}{\rho_{W'}(z)} \geq \frac{1}{C} \cdot \log\left(\frac{|z|}{R}\right), \qquad \text{for } \ |z| > M.$$

Hence, we have $\dfrac{\rho_V(z)}{\rho_W(z)} \to \infty$ as $z \to \infty$. Thus, the real valued function η is continuous on $\widetilde{V}$. Hence, η attains its infimum on the set $\widetilde{V}$. Then the claim folllows from (4.2) and (4.3). $\square$

In section 5.2, we will construct a semiconjugacy between the Julia set of a parabolic transcendental function f and the Julia set of a disjoint type function. To do so, we need a metric defined on the Julia set that is expanding. However, the hyperbolic metric ρ_W is not defined on the whole Julia set because $\text{Par}(f) \subset \mathcal{J}(f) \cap \partial W$ by Lemma 3.15 and Proposition 4.6(e). So the distance between a point $z \in W$ and any parabolic point with respect to the hyperbolic metric ρ_W is not defined. Therefore, we will modify the metric near any parabolic point of f. So we will define a new metric σ on W.

For $z \in \mathbb{C}$ let $\mathrm{d}_{\mathrm{Par}}(z)$ denote the distance from z to the finite set $\mathrm{Par}(f)$, i.e. $\mathrm{d}_{\mathrm{Par}}(z) := \inf\{|z - \zeta| \colon \zeta \in \mathrm{Par}(f)\}$. Now let us define a metric σ on W.

Fix $M > 0$ and $\epsilon > 0$. We define σ as follows

$$\sigma(z) := \begin{cases} \rho_W(z), & \text{for } \mathrm{d}_{\mathrm{Par}}(z) \geq \epsilon, \\ \min\{\rho_W(z), M\}, & \text{for } \mathrm{d}_{\mathrm{Par}}(z) < \epsilon. \end{cases} \tag{4.8}$$

Observe that σ depends on M and ϵ.

Recall that if ζ is a parabolic periodic point of f with period k and multiplier $\lambda = \exp(2\pi i p/q)$ where $(p, q) = 1$, then ζ is a parabolic fixed point for f^{kq} with multiplier one. So in the next theorem we will assume for simplicity that all parabolic points of f are fixed points with multiplier one.

Theorem 4.10. *Let $f \colon V \to W$ be as in Proposition 4.6. Suppose that each point in $\mathrm{Par}(f)$ is fixed and has multiplier one. Then there exist $M > 0$ and $\epsilon > 0$ such that the metric σ defined in (4.8) satisfies the following for all $0 < \delta < \epsilon$.*

(a) *There exists $\lambda = \lambda(\delta) > 1$ such that $\|\mathrm{D}f(z)\|_\sigma \geq \lambda$ for all $z \in V$ with $\mathrm{d}_{\mathrm{Par}}(z) \geq \delta$.*

(b) *$\|\mathrm{D}f(z)\|_\sigma \geq |f'(z)| > 1$ for all $z \in V$ with $\mathrm{d}_{\mathrm{Par}}(z) < \epsilon$.*

Proof. For each $\zeta \in \mathrm{Par}(f)$, let r_ζ be as in Lemma 4.7. Note that for any of the conditions in Lemma 4.7, if the condition holds for $r_\zeta > 0$ then it clearly holds for any $r \leq r_\zeta$.

Define $r := \min\{r_\zeta/2 \colon \zeta \in \mathrm{Par}(f)\}$. Then we make the following claim.

(C.1) Recall from Lemma 4.7 that $V_{\zeta,\eta}$ is the connected component of $f^{-1}(D(\zeta,\eta))$ that contains ζ. There exists $M>0$ such that $\rho_W(z)/|f'(z)| \leq M/2$ for all $z \in \bigcup_{\zeta\in\mathrm{Par}(f)} \left(f^{-1}(D(\zeta,r/2))\backslash V_{\zeta,r/2}\right)$.

To prove (C.1) let us first prove the following claim.

Claim (1). Suppose that $\zeta \in \mathrm{Par}(f)$ and ρ_W is the hyperbolic metric on W. Then there exists $M_\zeta > 0$ such that

$$\frac{\rho_W(z)}{|f'(z)|} \leq \frac{M_\zeta}{2},$$

for all $z \in f^{-1}\left(D(\zeta,r/2)\right)\backslash V_{\zeta,r/2}$.

Proof of claim. Set $Q := f^{-1}(D(\zeta,r))\backslash V_{\zeta,r}$, then it follows from Lemma 4.7 that $Q \subset W$ and the map $f: Q \to D(\zeta,r)$ is a covering map. Let $z \in Q$, then by Pick's theorem we have $\rho_Q(z) > \rho_W(z)$ and

$$1 = \|\mathrm{D}f(z)\|_Q^{D(\zeta,r)} = \frac{\rho_{D(\zeta,r)}(f(z))}{\rho_Q(z)} \cdot |f'(z)| < \frac{\rho_{D(\zeta,r)}(f(z))}{\rho_W(z)} \cdot |f'(z)|. \tag{4.9}$$

Note that the real-valued continuous function $\rho_{D(\zeta,r)}$ is bounded from above on $D(\zeta,r/2)$. So set

$$M_\zeta := 2\sup\{\rho_{D(\zeta,r)}(w)\colon w \in D(\zeta,r/2)\}.$$

It follows from (4.9) that

$$\frac{\rho_W(z)}{|f'(z)|} < \rho_{D(\zeta,r)}(f(z)) \leq M_\zeta/2, \qquad \text{for } z \in f^{-1}(D(\zeta,r/2))\backslash V_{\zeta,r/2}.$$

as claimed. △

Now, we set

$$M := \max_{\zeta\in\mathrm{Par}(f)}\{M_\zeta\}.$$

so that

$$\frac{\rho_W(z)}{|f'(z)|} \leq \frac{M}{2}, \qquad \text{for } z \in \bigcup_{\zeta} \left(f^{-1}(D(\zeta, r/2)) \backslash V_{\zeta, r/2} \right).$$

This proves (C.1).

We claim that the following holds.

Claim (2). Suppose that $0 < \epsilon \leq r/2$ and $z \in V$. If $\mathrm{d}_{\mathrm{Par}}(z) < \epsilon$ then $\min\{\rho_W(z), \rho_W(f(z))\} > 2M$, and if $\mathrm{d}_{\mathrm{Par}}(f(z)) < \epsilon$ then $\rho_W(f(z)) > 2M$.

Proof of claim. Let $z \in W$ and $\zeta \in \mathrm{Par}(f)$ then $\rho_W(z) \to \infty$ as $z \to \zeta$ because ζ is a finite boundary point of W. Hence we can choose $L_\zeta > 0$ sufficiently small such that $\rho_W(z) > 2M$ for $z \in D(\zeta, L_\zeta) \cap W$. Set $L := \min\{L_\zeta \colon \zeta \in \mathrm{Par}(f)\}$, thus we have $\rho_W(z) > 2M$ for $z \in D(\zeta, L)$ and all $\zeta \in \mathrm{Par}(f)$.
Let us choose $0 < \epsilon \leq \min\{r/2, L\}$ such that $f(D(\zeta, \epsilon)) \subset D(\zeta, L)$ for all $\zeta \in \mathrm{Par}(f)$. Thus if $z \in V$ and $\mathrm{d}_{\mathrm{Par}}(z) < \epsilon$ then there exists $\zeta \in \mathrm{Par}(f)$ such that $z \in D(\zeta, \epsilon) \cap V$. This implies that $z, f(z) \in D(\zeta, L) \cap W$ and hence $\rho_W(z) > 2M$ and $\rho_W(f(z)) > 2M$. Moreover, if $z \in V$ and $\mathrm{d}_{\mathrm{Par}}(f(z)) < \epsilon$ then $f(z) \in W$ and again there exists $\zeta \in \mathrm{Par}(f)$ such that $f(z) \in D(\zeta, \epsilon) \subseteq D(\zeta, L)$. Hence we have $\rho_W(f(z)) > 2M$. $\triangle$

Let ϵ and M be as above. We will prove now (a) and (b) in the statement.

Let $z \in V$ and $0 < \delta < \epsilon$. Then by definition

$$\|\mathrm{D}f(z)\|_\sigma = \frac{\sigma(f(z))}{\sigma(z)} \cdot |f'(z)|. \tag{4.10}$$

We will study the derivative $\|\mathrm{D}f(z)\|_\sigma$ by considering all the possible cases according to the distance between z or $f(z)$ and the set of parabolic points. Thus, we will consider the following cases.

i. If $d_{\mathrm{Par}}(z) \geq \epsilon$ and $d_{\mathrm{Par}}(f(z)) \geq \epsilon$, then by (4.8) and (4.10) we have $\|\mathrm{D}f(z)\|_\sigma = \|\mathrm{D}f(z)\|_W$. Hence by Lemma 4.9, we have

$$\mu := \inf_{z\in V, |z-\zeta|\geq\epsilon} \|\mathrm{D}f(z)\|_\sigma > 1.$$

for all $\zeta \in \mathrm{Par}(f)$.

ii. If $d_{\mathrm{Par}}(z) \geq \epsilon$ and $d_{\mathrm{Par}}(f(z)) < \epsilon$, then by (4.8) and (4.10) we have

$$\|\mathrm{D}f(z)\|_\sigma = \frac{\min\{\rho_W(f(z)), M\}}{\rho_W(z)} \cdot |f'(z)|.$$

It follows from Claim (2) that

$$\|\mathrm{D}f(z)\|_\sigma = M \cdot \frac{|f'(z)|}{\rho_W(z)}.$$

Observe that by the definition of d_{Par} there exists $\zeta \in \mathrm{Par}(f)$ such that $|z-\zeta| \geq \epsilon$ and $|f(z)-\zeta| < \epsilon$. Thus, by Lemma 4.7(3) we have $z \notin V_{\zeta, r_\zeta}$, and hence $z \notin V_{\zeta,2r}$. Hence, we have

$$z \in f^{-1}(D(\zeta,\epsilon))\backslash V_{\zeta,r} \subseteq f^{-1}(D(\zeta, r/2)\backslash V_{\zeta,r/2}.$$

Thus by (C.1) we have

$$\|\mathrm{D}f(z)\|_\sigma \geq M \cdot \frac{2}{M} = 2.$$

iii. Let $0 < d_{\mathrm{Par}}(z) < \epsilon$ and $d_{\mathrm{Par}}(f(z)) \geq \epsilon$. Then again by (4.8) and (4.10)

$$\|\mathrm{D}f(z)\|_\sigma = \frac{\rho_W(f(z))}{\min\{\rho_W(z), M\}} \cdot |f'(z)|.$$

It follows from Claim (2) that

$$\|\mathrm{D}f(z)\|_\sigma = \frac{\rho_W(f(z))}{M} \cdot |f'(z)| > \frac{2M}{M} \cdot |f'(z)| = 2|f'(z)|.$$

iv. Let $0 < \mathrm{d}_{\mathrm{Par}}(z) < \epsilon$ and $\mathrm{d}_{\mathrm{Par}}(f(z)) < \epsilon$. Using equations (4.8) and (4.10) once more, we have

$$\|\mathrm{D}f(z)\|_\sigma = \frac{\min\{\rho_W(f(z)), M\}}{\min\{\rho_W(z), M\}} \cdot |f'(z)|.$$

It follows from Claim (2) that

$$\|\mathrm{D}f(z)\|_\sigma = \frac{M}{M} \cdot |f'(z)| = |f'(z)|.$$

In the cases (iii) and (iv), we have $z \in D(\zeta, \epsilon) \cap W$ for some $\zeta \in \mathrm{Par}(f)$. Thus by Lemma 4.7(2), we have $|f'(z)| > 1$ for z in (iii) and (iv), which proves (b).

It follows from the cases (ii) and (iii) that

$\inf\{\|\mathrm{D}f(z)\|_\sigma \colon \mathrm{d}_{\mathrm{Par}}(z) \geq \epsilon, \mathrm{d}_{\mathrm{Par}}(f(z)) < \epsilon \text{ or } \delta \leq \mathrm{d}_{\mathrm{Par}}(z) < \epsilon, \mathrm{d}_{\mathrm{Par}}(f(z)) \geq \epsilon\} \geq 2.$

Hence, we can choose

$$\lambda := \min\{\mu, 2, \inf\{|f'(z)| \colon \delta \leq \mathrm{d}_{\mathrm{Par}}(z) < \epsilon, \mathrm{d}_{\mathrm{Par}}(f(z)) < \epsilon\}\} > 1,$$

which proves (a). □

We are going now to study the expansion property of a parabolic function with respect to the metric σ. For $n \in \mathbb{N}$, we will find a global lower bound of the derivative $\|Df^n(z)\|_\sigma$ at certain points z with $f^n(z) \in W$. By Proposition 4.6(d) this means that $f^j(z) = z_j \in V for 0 \leq j < n$. So we need to study the derivative at each point $z, f(z), \dots, f^n(z)$. Let $\delta > 0$ be as in Theorem 4.10. If $\mathrm{d}_{\mathrm{Par}}(z_j) \geq \delta$ for all $1 \leq j \leq n$ then it follows from Theorem 4.10(a) that there there exists $\lambda > 1$ which depends only on δ such that

$$\|Df^n(z)\|_\sigma \geq \lambda^n.$$

This gives uniform expansion for the function f with respect to the metric σ at certain points of W. However, this cannot be the case for all the points $z \in W$. Indeed, it is possible that the sequence $z, f(z), \dots, f^n(z)$ contains some points at which the derivative with respect to the metric σ is the Euclidean derivative. By Theorem 4.10, there exists $\epsilon > 0$ such that these points z_j satisfy that $\mathrm{d}_{\mathrm{Par}}(z_j) < \epsilon$. Hence, by Proposition 3.20 the function f may have less expansion than the previous case. We will study this expansion property for such a function in the next two results.

Proposition 4.11. *Let $f \colon V \to W$ be as in Proposition 4.6, such that all parabolic points of f are fixed and with multiplier one. Let σ be the metric defined in (4.8). Then for all $\ell > 0$ there exist $C > 0$ and $a > 1$ with the following property. If $n \in \mathbb{N}$ and $z \in V$ such that $f^n(z) \in W$ and $\mathrm{d}_{\mathrm{Par}}(f^n(z)) > \ell$, then*

$$\|Df^n(z)\|_\sigma \geq C\; n^a.$$

Proof. For $z \in V$ and $j \geq 0$ we set $z_j := f^j(z)$, then we have

$$\|Df^n(z)\|_\sigma = \prod_{j=1}^{n} \|Df(z_{n-j})\|_\sigma. \tag{4.11}$$

For each $\zeta \in \mathrm{Par}(f)$, let us choose $K_\zeta > 0$ and a well-behaved repelling petal $\mathcal{P}$ for each repelling vector at ζ according to Proposition 3.20. Let $\mathcal{F}_{\mathrm{R}}(\zeta)$ be the union of these petals. Let $r_\zeta > 0$ be as in Proposition 4.6(b). Set

$$K := \min\left\{\min_{\zeta\in\mathrm{Par}(f)}\{K_\zeta\}, 1\right\},$$

and

$$\widetilde{r} := \min_{\zeta \in \mathrm{Par}(f)} \{r_\zeta\}.$$

Let $\epsilon > 0$ be as in Theorem 4.10(b) and let $\ell > 0$. Choose $r := \min\{\widetilde{r}, \epsilon, \ell, 1\}$. This implies that Theorem 4.10(b) holds for all z with $\mathrm{d}_{\mathrm{Par}}(z) < r$. Note that each petal in $\mathcal{F}_{\mathrm{R}}(\zeta)$ makes an angle $\pi/(2p_\zeta) < \theta \leq \pi/p_\zeta$ at ζ. Since $r \leq \widetilde{r}$ then it follows from Proposition 4.6(b) that each component of $D(\zeta, r) \cap W$ is contained in one petal in the union $\mathcal{F}_{\mathrm{R}}(\zeta)$.

Suppose that f has multiplicity $p_\zeta + 1$ at ζ. Set $p := \max_{\zeta \in \mathrm{Par}(f)} \{p_\zeta\}$ and $a := (1+p)/p$. Define $\alpha_n := Krn^a$ for $n \in \mathbb{N}$. Then we choose $n_0 \in \mathbb{N}$ such that

$$\alpha_k \cdot \alpha_m \geq \alpha_{k+m}, \tag{4.12}$$

for all $k, m \geq n_0$.

By continuity of the iterates of f there exists $0 < \delta < r$ such that

$$\mathrm{d}_{\mathrm{Par}}(f^j(z)) < r, \quad \text{whenever} \quad \mathrm{d}_{\mathrm{Par}}(z) < \delta \quad \text{and} \quad j \leq n_0.$$

By Theorem 4.10(a), there exists $\lambda > 1$ such that the σ-derivative of f at every point z with $\mathrm{d}_{\mathrm{Par}}(z) \geq \delta$ satisfies

$$\|Df(z)\|_\sigma \geq \lambda. \tag{4.13}$$

Let us choose $C > 0$ such that

$$\lambda^m \geq Cr^{-1}K^{-1}(1+m)^a, \quad m \in \mathbb{N}. \tag{4.14}$$

Now, let $z \in V$ such that $f^n(z) \in W$ and $\mathrm{d}_{\mathrm{Par}}(z_n) > \ell$. Then we can decompose the orbit of z as follows. First, we call $B \subset \{0, \ldots, n-1\}$ a block if B has the following properties.

1. $B := \{j, j+1, \ldots, j+m-1\}$ for some $j, m \in \mathbb{N}$;

2. $\mathrm{d}_{\mathrm{Par}}(z_i) < r$ for all $i \in B$, and B is maximal with this property.

We say that B starts with j and has length m.

Let $B_1, B_2, \ldots, B_s$ for some $0 \le s < n$ be blocks of length at least n_0. For $k = 1, \ldots, s$, let the block B_k start at j_k and have length $m_k \ge n_0$. By definition, the points $z_{j_k}, \ldots z_{j_k+m_k-1}$ satisfy that there is a parabolic point ζ such that these points belong to $D(\zeta, r) \cap W$ for $k = 1, \ldots, s$. By our choice of r, the points $z_{j_k}, \ldots, z_{j_k+m_k-1}$ belong to $\mathcal{F}_{\mathrm{R}}(\zeta)$. It then follows from Proposition 3.20 and Theorem 4.10(b) that

$$\|Df^{m_k}(z_{j_k})\|_\sigma \ge |\,(f^{m_k})'\,(z_{j_k})| \ge K\ m_k^a\ |z_{j_k+m_k} - \zeta|.$$

By our choice of δ, if $0 \le j \le n$ and $j \notin [j_k, j_k + m_k]$ for all $k = 1, \ldots, s$ then $\mathrm{d}_{\mathrm{Par}}(z_j) \ge \delta$. Also, recall that $\mathrm{d}_{\mathrm{Par}}(z_n) > \ell \ge r$. Thus, we have

$$\mathrm{d}_{\mathrm{Par}}(z_{j_k+m_k}) \ge r, \quad k = 1, \ldots, s.$$

and hence

$$\|Df^{m_k}(z_{j_k})\|_\sigma \ge K r m_k^a. \tag{4.15}$$

Set $n_1 := \sum_{k=1}^{s} m_k$, and let $n_2 = n - n_1$ be the number of the values j that are not in any of the blocks B_k. It follows from (4.11) that

$$\|Df^n(z)\|_\sigma = \prod_{j \notin B_j} \|Df(z_j)\|_\sigma \cdot \prod_{k=1}^{s} \|Df^{m_k}(z_{j_k})\|_\sigma. \tag{4.16}$$

By (4.13) and (4.15), we have

$$\|Df^n(z)\|_\sigma \ge \lambda^{n_2} \cdot \prod_{k=1}^{s} \alpha_{m_k}.$$

Let us assume that $n_1 \neq 0$. Since $m_k \geq n_0$ for all k, then it follows from (4.12) and (4.14) that

$$\begin{aligned}\|Df^n(z)\|_\sigma &\geq \lambda^{n_2} \cdot \alpha_{n_1} = \lambda^{n_2} \cdot \alpha_{n-n_2} \geq Cr^{-1}K^{-1}(1+n_2)^a Kr(n-n_2)^a \\ &= ((1+n_2)(n-n_2))^a = C\left[n + n_2\left(n-(n_2+1)\right)\right]^a \\ &\geq Cn^a.\end{aligned}$$

Note that $n + n_2\left(n-(n_2+1)\right) > n$ as $n > n_2$.

Now, if $n_1 = 0$ then $n_2 = n$ and it follows from (4.14) and (4.16) that

$$\|Df^n(z)\|_\sigma \geq \lambda^n \geq Cr^{-1}K^{-1}(1+n)^a > Cn^a.$$

This gives the required lower bound. □

Proposition 4.12. *Let $f\colon V \to W$ be as in Proposition 4.6. Let σ be the metric defined in (4.8). Then there exist $R > 0$, $C > 0$ and $a > 1$ with the following property. If $n \in \mathbb{N}$ and $z \in V$ such that $f^n(z) \in W$ and $\mathrm{d}_{\mathrm{Par}}(f^n(z)) > R$, then*

$$\|Df^n(z)\|_\sigma \geq C\, n^a.$$

Proof. It suffices to prove the lower bound on $\|Df^n(z)\|_\sigma$ holds for all sufficiently large n, because then it holds for all n by adjusting the constant C. Let $n \in \mathbb{N}$ and $0 < k < n$ be a common multiple of the periods of all parabolic periodic points of f. Let $0 < q < n$ be such that the multiplier λ_ζ at each parabolic point ζ satisfies that $\lambda_\zeta^q = 1$. Then, all parabolic periodic points of the function f are parabolic fixed points of f^{kq} with multiplier one.

Set $s := kq$. Let $\epsilon > 0$ be as in Theorem 4.10, and let us choose $R > 0$ such that

$$f^{-j}\{\mathbb{C}\backslash\overline{D(0,R)}\} \subset \{z \in \mathbb{C}\colon \mathrm{d}_{\mathrm{Par}}(z) \geq \epsilon\}, \quad j = 0, \ldots, s. \tag{4.17}$$

Now, let $z \in W$ such that $\mathrm{d}_{\mathrm{Par}}(f^n(z)) > R$. Then, there exist $1 \leq r < n$ and $0 < m < s$ such that

$$f^n(z) = f^m\left(f^{rs}(z)\right). \tag{4.18}$$

Then it follows that

$$\|Df^n(z)\|_\sigma = \|Df^m\left(f^{rs}(z)\right)\|_\sigma \cdot \|Df^{rs}(z)\|_\sigma.$$

Since $m < s$ and $\mathrm{d}_{\mathrm{Par}}(f^n(z)) > R$ then it follows from (4.17) that

$$\mathrm{d}_{\mathrm{Par}}(f^{rs}(z)) \geq \epsilon$$

Thus, by definition

$$\|Df^m\left(f^{rs}(z)\right)\|_\sigma = \|Df^m\left(f^{rs}(z)\right)\|_W.$$

It follows from Theorem 4.10(a) that

$$\|Df^m\left(f^{rs}(z)\right)\|_\sigma > 1.$$

Hence, we have

$$\|Df^n(z)\|_\sigma > \|Df^{rs}(z)\|_\sigma.$$

By Proposition 4.11, there exist $\widetilde{C} > 0$ and $a > 1$ such that

$$\|Df^n(z)\|_\sigma > \widetilde{C} r^a = \widetilde{C}\left((n-m)/s\right)^a.$$

Since $n = m + rs$, $r \geq 1$ and $s > m$ then $n > 2m$, and hence $n - m > n/2$. Hence, we have

$$\|Df^n(z)\|_\sigma > \widetilde{C}\left(n/2s\right)^a.$$

By choosing $C := \widetilde{C}/(2s)^a$, the claim follows. $\square$

Chapter 5

Functions of disjoint type and Semiconjugated Julia sets

5.1 Functions of disjoint type

The dynamics of disjoint type functions was described by several results, see for example [MB12, Propositions 2.8 and 2.9], [BJR12] and [RG16]. A key property of disjoint type functions is that for every $f \in \mathcal{B}$ there exists $\lambda > 0$ such that $g(z) := f(\lambda z)$ is of disjoint type, see [Rem09, 261] and [Six17, Lemma 7.1]. By using this fact, we will be able to transfer some of the dynamical properties of the disjoint type function g to the dynamics of f.

We will give below the definition of a function of disjoint type and some crucial results for this class of functions. We will first give the definition of a hyperbolic function.

Definition 5.1. *An entire function f is* hyperbolic *if and only if $P(f)$ is a compact subset of the Fatou set of f.*

Definition 5.2. *A transcendental entire function $f \in \mathcal{B}$ is of* disjoint type *if it is hyperbolic and the Fatou set $\mathcal{F}(f)$ is connected.*

The next results give more properties of functions of disjoint type.

Proposition 5.3. [MB12, Proposition 2.8] *Suppose that $f \in \mathcal{B}$. Then the following statements are equivalent:*

(a) *The map f is of disjoint type.*

(b) *f has a unique attracting fixed point and $P(f)$ is a compact subset of its immediate basin of attraction.*

(c) *There exists a Jordan domain $D \supset S(f)$ such that $\overline{f(D)} \subset D$.*

Proposition 5.4. [MB12, Proposition 2.9] *Suppose that f is of disjoint type function. Then $I(f)$ is disconnected.*

The next theorem concerns functions of disjoint type with finite order. We say that a transcendental entire function f has *finite order* if

$$\log\log|f(z)| = O\left(\log|z|\right), \quad \text{as} \quad |z| \to \infty.$$

A subset $A \subset \mathbb{C}$ is called a *Cantor bouquet* if it is ambiently homeomorphic to a straight brush in the sense of [AO93]. We say that $A, B \subset \mathbb{C}$ are *ambiently homeomorphic* if there exists a homeomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $\phi(A) = B$.

Theorem 5.5. [BJR12, Theorem 1.5] *Suppose that f is a disjoint type function of finite order or can be written as a finite composition of finite order functions in the class $\mathcal{B}$. Then the Julia set $\mathcal{J}(f)$ is a Cantor bouquet.*

5.2 The existence of semiconjugacies

Let f be a parabolic transcendental entire function. In this section we will consider all the notions in Chapter 4. We are going to construct a continuous surjection

$$\phi : \mathcal{J}(g) \to \mathcal{J}(f),$$

where $g \in \{f(\lambda z) \colon \lambda \in \mathbb{C}\}$ is a disjoint type function and such that

$$f \circ \phi(z) = \phi \circ g(z), \tag{5.1}$$

for all $z \in \mathcal{J}(g)$.

Let $R > 0$ be as in Proposition 4.12. Choose $K \geq 2R$ sufficiently large such that

$$\overline{U} \subset D(0, K/2). \tag{5.2}$$

Then we choose $L \geq K$ such that

$$f^{-1}\left(\mathbb{C} \backslash D(0, L)\right) \subset \mathbb{C} \backslash D(0, K+1). \tag{5.3}$$

Set $M := K/L \leq 1$ and $g(z) := f(Mz)$.

We will now prove that the function g is disjoint type. It follows from (5.3) that

$$g^{-1}\left(\mathbb{C} \backslash D(0, L)\right) \subset \mathbb{C} \backslash D(0, L+1).$$

which implies that

$$g^{-1}\left(\mathbb{C} \backslash D(0, L)\right) \subset \mathbb{C} \backslash \overline{D(0, L)}.$$

Thus, by continuity

$$\overline{g\left(D(0, L)\right)} \subset D(0, L). \tag{5.4}$$

We will show that $S(g) = S(g)$. If $\widetilde{s}$ is a critical value of g then there exists $z \in \mathbb{C}$ such that $g'(z) = 0$ and $g(z) = \widetilde{s}$. Since $g(z) = f(Mz)$ then $g'(z) = Mf'(Mz) = 0$. This means that $f'(Mz) = 0$, and hence Mz is a critical point of f and $f(Mz) = \widetilde{s}$ is a critical value of f. If $\widetilde{a}$ is an asymptotic value of g then there exists a curve $\widetilde{\gamma} = \widetilde{\gamma}(t)$ such that $\widetilde{\gamma}(t) \to \infty$ as $t \to \infty$ and $\lim_{t\to\infty} g(\widetilde{\gamma}(t)) = \widetilde{a}$. If we take the curve $\gamma := M\widetilde{\gamma}$ then we have $\gamma(t) \to \infty$ as $t \to \infty$ and $\lim_{t\to\infty}\left(f\left(\gamma(t)\right)\right) = \lim_{t\to\infty}\left(f\left(M\widetilde{\gamma}(t)\right)\right) = \lim_{t\to\infty}\left(g\left(\widetilde{\gamma}(t)\right)\right) = \widetilde{a}$.

Hence, $\widetilde{a}$ is an asymptotic value of f. This proves that $S(g) = S(g)$.

Note that $S(f) \subset \overline{U}$ by Proposition 4.6(a). It follows from (5.2) that $S(f) \subset D(0, K/2)$. Hence, we have

$$S(g) \subset D(0, L/2) \tag{5.5}$$

Then, it follows from (5.4) and Proposition 5.3 that g is a disjoint type function.

For $j \geq 0$ we define

$$V_j := f^{-j}\left(\mathbb{C}\backslash\overline{D(0,K)}\right) \quad \text{and} \quad U_j := g^{-j}\left(\mathbb{C}\backslash\overline{D(0,L)}\right).$$

It is easy to see that $U_{j+1} \subset U_j \subset W$ for all $j \geq 0$.

Remark 5.6. *By Montel's Theorem the disc $D(0, L)$ is contained in a component of $\mathcal{F}(g)$. This implies that $\mathcal{J}(g)$ is the set of those points which are never mapped into $\overline{D(0, L)}$. Hence, we can say that*

$$\mathcal{J}(g) = \bigcap_{j\geq 0} U_j.$$

We will construct a sequence of conformal isomorphisms ϕ_j such that $\phi_0 \equiv id$ and

$$\phi_j \colon U_{j-1} \to V_{j-1}, \quad \text{for} \ \ j \geq 1.$$

and it satisfies

$$f \circ \phi_{j+1} = \phi_j \circ g, \quad \text{for} \ \ j \geq 0. \tag{5.6}$$

We will define ϕ_j inductively. By definition $\phi_1(z) = Mz$. Note that $U_0 \subset V_0$. So for each $z \in U_0$ we will choose $\gamma_1(z) \subset V_0$ to be the line segment connecting $z = \phi_0(z)$ and $Mz = \phi_1(z)$. Observe here that by definition $Mz \in V_0$. To define ϕ_2 note first that if $z \in U_1$ then by definition $g(z) \in U_0$ and hence $Mg(z) \in V_0$. So for each $z \in U_1$ the line segment $\gamma_1(g(z))$ between $g(z)$ and $Mg(z)$ is contained in V_0. Since $f(\phi_1(z)) = \phi_0(g(z)) = g(z)$ by definition, then $\gamma_1(g(z))$ has a preimage component under f in V_1, say $\gamma_2(z)$, with end point at $\phi_1(z)$. We define $\phi_2(z)$ to be the other end point of $\gamma_2(z)$.

We proceed the construction inductively. Suppose that $\phi_j \colon U_{j-1} \to V_{j-1}$ is defined and equation (5.6) holds for j. This means that the curve $\gamma_j(z) \subset V_{j-1}$ is defined for all $z \in V_{j-1}$. So for each $z \in U_j$, we take the curve $\gamma_j(g(z)) \subset V_{j-1}$ between $\phi_{j-1}(g(z))$ and $\phi_j(g(z))$. By definition, we have that

$$f^{-1}\left(\gamma_j(g(z))\right) \subset V_j.$$

Since $f(\phi_j(z)) = \phi_{j-1}(g(z))$ then $\gamma_j(g(z))$ has a preimage component under f, say $\gamma_{j+1}(z)$, with end point at $\phi_j(z)$. Then we define $\phi_{j+1}(z)$ to be the other end point of $\gamma_{j+1}(z)$.

By definition the functions ϕ_j are continuous for all $j \geq 0$, as f and g are continuous functions. Moreover, each map ϕ_j is a conformal isomorphism from a component of U_{j-1} to a component of V_{j-1}, by construction .

Theorem 5.7. *Let $f \colon V \to W$ be as in Proposition 4.6. Let σ be the metric defined on W in (4.8), and let $M \in \mathbb{C}$ be such that $g(z) := f(Mz)$ is of disjoint type. The maps $\phi_j|_{\mathcal{J}(g)}$ converge uniformly with respect to the metric σ to a continuous surjection*

$$\phi \colon \mathcal{J}(g) \to \mathcal{J}(f),$$

such that $f \circ \phi = \phi \circ g$. Moreover, $\phi \colon I(g) \to I(f)$ is a homeomorphism.

Proof. Let $\mathrm{d}_\sigma(w_1, w_2)$ denote the distance between $w_1, w_2 \in W$ with respect to the metric σ, and let $\ell_\sigma(\gamma)$ denote the $\sigma-$length of a curve $\gamma \subset W$. Recall that $U_j \subset U_{j-1}$ then by definition the maps ϕ_j and ϕ_{j+1} are both defined in a neighbourhood of $z \in W$. By construction the curve $\gamma_{j+1}(z)$ is connecting the points $\phi_j(z)$ and $\phi_{j+1}(z)$. Thus, by definition

$$\mathrm{d}_\sigma(\phi_{j+1}(z), \phi_j(z)) \leq \ell_\sigma(\gamma_{j+1}(z)) \tag{5.7}$$

Suppose that $z \in U_0$ then by construction and (5.2)

$$\gamma_1(z) \subset V_0 = \mathbb{C} \backslash \overline{D(0,K)} \subset \mathbb{C} \backslash \overline{D(0,K/2)} \subset W.$$

We are going to find a uniform bound on the length $\ell_\sigma(\gamma_1(z))$. So we can estimate the length $\ell_\sigma(\gamma_1(z))$ using the hyperbolic metric on $\widetilde{V} := \mathbb{C} \backslash \overline{D(0,K/2)}$. By Pick's Theorem

$$\ell_\sigma(\gamma_1(z)) \leq \ell_{\widetilde{V}}(\gamma_1(z)).$$

Recall from (2.5) that the hyperbolic density on $\widetilde{V}$ is given by

$$\rho_{\widetilde{V}}(z) = \frac{1}{|z|\,(\log|z| - \log(K/2))}.$$

Since $\gamma_1(z)$ is the line segment connecting the points z and Mz then, we have

$$\begin{aligned}
\ell_{\widetilde{V}}(\gamma_1(t)) &= \int_0^1 \frac{|z|(M-1)}{|z||(M-1)t+1|\,(\log(|z||(M-1)t+1|) - \log(K/2))}\, dt \\
&= -\left[\log\left(\log(|z||(M-1)t+1|) - \log(K/2)\right)\right]_0^1 \\
&= \log\left(\frac{\log|z| - \log(K/2)}{\log(M|z|) - \log(K/2)}\right) = \log\left(1 + \frac{\log(1/M)}{\log M|z| - \log(K/2)}\right) \\
&= \log\left(1 + \frac{\log(1/M)}{\log(2M|z|/K)}\right) = \log\left(1 + \frac{\log(1/M)}{\log(2|z|/L)}\right).
\end{aligned}$$

Since $z \in U_0$ then $|z| > L$. Hence, we have

$$\ell_\sigma(\gamma_1(z)) \leq \ell_{\widetilde{V}}(\gamma_1(t)) \leq \log\left(1 + \frac{\log(1/M)}{\log 2}\right) := \mu. \tag{5.8}$$

Note that $\gamma_{j+1}(z) \subset V_j \subset W$ is obtained as a pullback of $\gamma_1(g^j(z))$ under the map f^j. Note also that if $w \in \gamma_1(g^j(z)) \subset V_0$ then $|w| > K$. It follows from Proposition 4.6(e) and (5.2) that $\mathrm{Par}(f) \subset D(0, K/2)$ and hence $\mathrm{d}_{\mathrm{Par}}(w) > K/2 > R$. Moreover, each pullback satisfies that $\gamma_k(g^j(z)) \subset V_{k-1} \subset W$ for all $0 < k \leq j$. It follows from Proposition 4.12, together with (5.7) and (5.8) that

$$\mathrm{d}_\sigma(\phi_{j+1}(z), \phi_j(z)) \leq \frac{\mu}{Cj^a}, \tag{5.9}$$

where $C > 0$ and $a > 1$. This implies that the maps $\phi_j|_{\mathcal{J}(g)}$ form a Cauchy sequence which converges to a continuous limit function ϕ and, by Remark 5.6

$$\phi\colon \mathcal{J}(g) \to W.$$

By (5.9) and by construction, the function ϕ satisfies

$$\mathrm{d}_\sigma(\phi(z), z) \leq \sum_{j=0}^{\infty} \mathrm{d}_\sigma(\phi_{j+1}(z), \phi_j(z)) \leq \sum_{j=1}^{\infty} \mu . \frac{1}{Cj^a} := \alpha, \quad z \in \mathcal{J}(g). \tag{5.10}$$

and

$$f^n(\phi(z)) = \phi(g^n(z)), \quad n \in \mathbb{N},\ z \in \mathcal{J}(g). \tag{5.11}$$

Suppose that $\phi(z) \in \phi\left(I(g)\right)$. Then $g^n(z) \to \infty$ as $n \to \infty$. It follows from (5.10) that

$$\phi\left(g^n(z)\right) \to \infty \quad \text{as} \quad n \to \infty.$$

Then, by equation (5.11)

$$f^n\left(\phi(z)\right) \to \infty \quad \text{as} \quad n \to \infty.$$

Hence, $\phi(z) \in I(f)$ which implies that

$$\phi(I(g)) \subset I(f).$$

We will now prove that $\phi\colon I(g) \to I(f)$ is surjective. Let $w \in I(f)$ then there exists $N \in \mathbb{N}$ such that $|f^j(w)| > K$ for $j > N$, which means that $w \in V_j$ for $j > N$. Set $z_j := \phi_j^{-1}(w)$. If z is an accumulation point of the sequence $(z_j)_{j\geq 0}$ then by continuity $z = \phi(w)$, and z must be finite by (5.10). This proves that $\phi\colon I(g) \to I(f)$ is surjective.

Now, we will show that $\phi\colon I(g) \to I(f)$ is injective. Recall from (5.5) that $S(g) \subset D(0, L)$, then $g^{-1}(U_0) = \mathbb{C}\backslash g^{-1}\left(\overline{D(0, L)}\right)$ is a countable union of simply connected unbounded components (tracts) T_i. Let γ be a curve connecting $\partial D(0, L)$ to ∞ such that $\gamma \cap T_i = \emptyset$ for all i. Then, the preimages of γ under f split the tracts into simply connected fundamental domains $\mathcal{F}_i$. For the sake of contradiction, let $z_1, z_2 \in I(g)$ such that $z_1 \neq z_2$ and $\phi(z_1) = \phi(z_2)$. We will prove that there exists $k \in \mathbb{N}$ sufficiently large such that $g^k(z_1)$ and $g^k(z_2)$ lie in the same fundamental domain $\mathcal{F}_k$. By construction, there exists a curve $\widetilde{\gamma}_n$ connecting $g^n(z_1)$ to $\phi((g^n(z_1))$ and a curve $\widetilde{\widetilde{\gamma}}_n$ connecting $g^n(z_2)$ to $\phi\left(g^n(z_2)\right)$ for all $n \in \mathbb{N}$. Let γ_n be the union of $\widetilde{\gamma}_n$ and $\widetilde{\widetilde{\gamma}}_n$, then by construction

$$\gamma_n \subset f(\gamma_{n-1}), \quad \text{for} \;\; n \in \mathbb{N}. \tag{5.12}$$

Note that the curve γ_n has a uniformly bounded length by (5.10). Thus, we can choose $k \in \mathbb{N}$ sufficiently large such that the curves γ_{k+1} and γ_{k+2} do not intersect with the disc $D(0, L/2)$ that contains the set of singular values of g. It follows from (5.12) that there exists a branch F of the inverse of f which maps $f(\gamma_{k+1})$ to γ_{k+1}. Then, it follows from the definition of g that there exists a branch G of the inverse of g such that $G(w) = (1/M)F(w)$. Hence, we have

$$G\left(g^{k+2}(z_1)\right) = \frac{1}{M}F\left(g^{k+2}(z_1)\right) = \frac{1}{M}MG\left(g^{k+2}(z_1)\right) = g^{k+1}(z_1),$$

and similarly for z_2. Thus, we can deduce now that $g^{k+1}(z_1)$ and $g^{k+1}(z_2)$ belong to the same tract. By repeating the argument we can prove that $g^k(z_1)$ and $g^k(z_2)$ belong to the same fundamental domain $\mathcal{F}_k$ for sufficiently large k.

Since $\phi(z_1) = \phi(z_2)$ then it follows from equation (5.11) that

$$\phi\left(g^k(z_1)\right) = f^k\left(\phi(z_1)\right) = f^k\left(\phi(z_2)\right) = \phi\left(g^k(z_2)\right).$$

It follows then from (5.10) that

$$\mathrm{d}_\sigma\left(g^k(z_1), g^k(z_2)\right) \leq \mathrm{d}_\sigma\left(g^k(z_1), \phi\left(g^k(z_1)\right)\right) + \mathrm{d}_\sigma\left(g^k(z_2), \phi\left(g^k(z_1)\right)\right) \leq 2\alpha,$$

This equation shows that there are points in the orbits of z_1 and z_2 stay a bounded σ-distance, and hence a bounded Euclidean distance apart, under iteration of g. However, a result of Rempe-Gillen [Rem09, Lemma 2.8] states that there is a logarithmic transform $\widetilde{g}$ of g, and two points with the same address have orbits for which the Euclidean distance grows exponentially. We can deduce a contradiction. Hence, ϕ is injective on $I(g)$.

We will show now that $\phi\colon \mathcal{J}(g) \to \mathcal{J}(f)$ is a continuous surjection. It follows from (5.10) that

$$\phi\left(g^n(z)\right) \to \infty \quad \text{if and only if} \quad g^n(z) \to \infty. \tag{5.13}$$

Thus, ϕ can be extended to a continuous function $\widetilde{\phi}\colon \mathcal{J}(g) \cup \{\infty\} \to \mathcal{J}(f) \cup \{\infty\}$ where $\widetilde{\phi}\left(\mathcal{J}(g)\right) = \phi\left(\mathcal{J}(g)\right)$ and $\widetilde{\phi}(\infty) = \infty$. Since $\widetilde{\phi}$ is continuous and $\mathcal{J}(g) \cup \{\infty\}$ is a compact set then the set $\widetilde{\phi}\left(\mathcal{J}(g) \cup \{\infty\}\right)$ is compact, and hence the set $\phi\left(\mathcal{J}(g)\right)$ is closed. Since the function g is in class $\mathcal{B}$ then it follows from (5.11), the property (5.13) and Theorem 2.9 that

$$I(f) \subset \phi\left(\mathcal{J}(g)\right) = \phi\left(\overline{I(g)}\right) \subset \overline{\phi\left(I(g)\right)} = \overline{I(f)}.$$

Since $\phi\left(\mathcal{J}(g)\right)$ is closed then it follows that $\overline{I(f)} \subset \overline{\phi\left(\mathcal{J}(g)\right)} = \phi\left(\mathcal{J}(g)\right)$. Hence, $\phi\left(\mathcal{J}(g)\right) = \overline{I(f)} = \mathcal{J}(f)$ as f is in class $\mathcal{B}$.

Let $U \subset I(g)$ be open. Since the set $\mathcal{J}(g) \cup \{\infty\}$ is compact then the set $\left(\mathcal{J}(g) \cup \{\infty\}\right) \backslash U$ is also compact. Since $\phi(I(g)) = I(f)$ then it follows from the continuity of the function $\widetilde{\phi}$ that $\left(\mathcal{J}(f) \cup \{\infty\}\right) \backslash \phi\left(U\right)$ is a compact set, which implies that $\phi(U)$ is open. Hence, $\phi\colon I(g) \to I(f)$ has a continuous inverse. This proves that $\phi\colon I(g) \to I(f)$ is a homeomorphism. □

The following results are consequences of Theorem 5.7.

Corollary 5.8. *Let $f \in \mathcal{B}$ be parabolic. Then the set $I(f)$ is disconnected.*

Proof. By Theorem 5.7, there exists a function g of disjoint type and homeomorphism ϕ such that $\phi\left(I(g)\right) = I(f)$. Hence, it follows from Theorem 5.4 that $I(f)$ is disconnected. □

A *pinched Cantor bouquet* is a subset of $\mathbb{C}$ that is ambiently homeomorphic to the quotient of a straight brush by a closed equivalence relation on its endpoints.

Corollary 5.9. *Suppose that $f \in \mathcal{B}$ is a parabolic function of finite order or can be written as a finite composition of finite order functions in the class $\mathcal{B}$. Then the Julia set $\mathcal{J}(f)$ is a pinched Cantor bouquet.*

Proof. By Theorem 5.7, there exists a function g of disjoint type and finite order, and a continuous surjection $\phi\colon \mathcal{J}(g) \to \mathcal{J}(f)$ such that $f\left(\phi(z)\right) = \phi\left(g(z)\right)$ for all $z \in \mathcal{J}(g)$. Moreover, ϕ is injective when restricted to $I(g)$ and $\phi\left(I(g)\right) = I(f)$. Then it follows from Theorem 5.5 that the Julia set of g is a Cantor bouquet. Since ϕ is not injective on $\mathcal{J}(g)$ and the only points in $\mathcal{J}(g)$ which are not in $I(g)$ are endpoints by [RRRS11, Theorem 4.7] then the claim follows. $\square$

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